Triangles on Curves

Problem 9

Yimu Mao

December 7, 2024

Notation

Notation: *Second-order-curve*

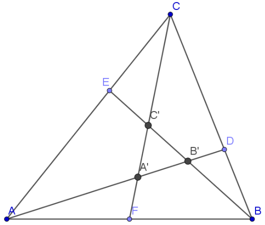
The three points of a triangle ΔABC are on a second-order-curve $\mathcal{R}$ of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

with $a^2 + b^2 + c^2 \neq 0$. While A and B are fixed, C can move freely along the curve.

Remark The Graph is a **parabola**, an **elipse** or a **hyperbola**.

Preliminary Lemma

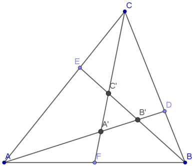


Lemma 1: *Inner Points $A'(x_1', y_1'), B'(x_2', y_2'), C'(x_3', y_3')$*

$$A(x_1, y_1), B(x_2, y_2), C(x_3, y_3), \quad \lambda_1 = \frac{|\overline{AE}|}{|\overline{AC}|}, \lambda_2 = \frac{|\overline{CD}|}{|\overline{DB}|}, \lambda_3 = \frac{|\overline{BB'}|}{|\overline{B'E}|}$$

$$\Rightarrow x_i' = \frac{x_i + \lambda_i x_{i+1} + \lambda_i \lambda_{i+1} x_{i+2}}{1 + \lambda_i + \lambda_i \lambda_{i+1}}, y_i' = \frac{y_i + \lambda_i y_{i+1} + \lambda_i \lambda_{i+1} y_{i+2}}{1 + \lambda_i + \lambda_i \lambda_{i+1}}$$

Proof Preliminary Lemma

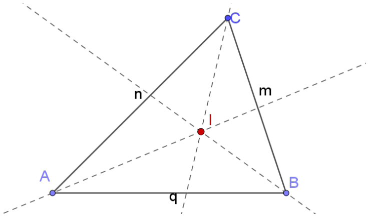


Lemma 1: *Proof via Menelaus's Theorem*

$$\frac{\overline{AE}}{\overline{AC}} \cdot \frac{\overline{CD}}{\overline{DB}} \cdot \frac{\overline{BB'}}{\overline{B'E}} = 1$$

Remark Menelaus's Theorem can be proven with the intercept theorem

Question 1. The Incentre

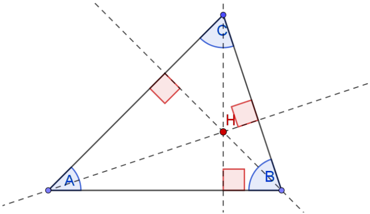


Coordinate of the incentre I

$$I\left(\frac{mx_1+nx_2+qx_3}{m+n+q},\frac{my_1+ny_2+qy_3}{m+n+q}\right)$$

proof idea: Lemma1 with $\lambda_1=\frac{n}{m},\lambda_2=\frac{q}{n}\lambda_3=\frac{m}{q}$ and $\overline{BC}=m,\overline{AC}=n,\overline{AB}=q$.

Question 2. The Orthocentre

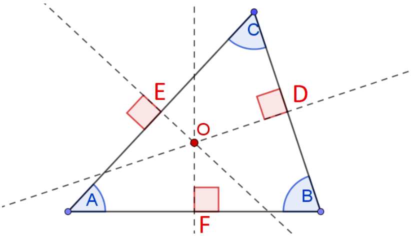


Coordinate of the Orthocentre H

$$H\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

proof idea: Lemma 1 with $\lambda_1 = \frac{\cot A}{\cot B}, \lambda_2 = \frac{\cot B}{\cot C}, \lambda_3 = \frac{\cot C}{\cot A}$

Question 3. The Circumcentre

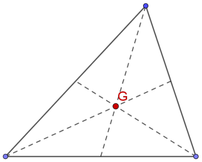


Coordinate of the Circumcentre O

$$O\left(\frac{x_1(\tan B+\tan C)+x_2(\tan C+\tan A)+x_3(\tan A+\tan B)}{2(\tan A+\tan B+\tan C)},\ \frac{y_1(\tan B+\tan C)+y_2(\tan C+\tan A)+y_3(\tan A+\tan B)}{2(\tan A+\tan B+\tan C)}\right)$$

proof idea: Lemma 1, O is Orthocenter of $\triangle DEF$

Question 4. The Centre of Gravity G

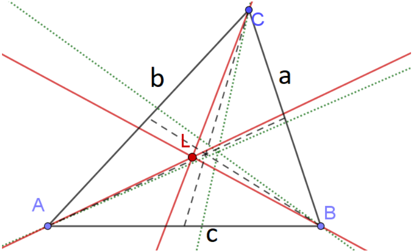


Coordinate of the Centre of Gravity

$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

proof idea: Lemma 1 and intersection point of medians

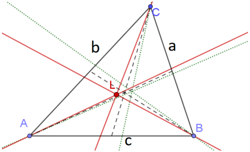
Question 5. The Intersection of the Symmedians



Definition of Symmedian

The three symmedian lines of a triangle are created by reflecting the bisectors on their corresponding median lines. They meet at the intersection point L.

Question 5. The Intersection of the Symmedians Coordinate



Coordinate of the Intersection of Symmedians L

$$L\left(\frac{\frac{a^2}{b^2}x_1+\frac{b^2}{c^2}x_2+\frac{c^2}{a^2}x_3}{\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}},\frac{\frac{a^2}{b^2}y_1+\frac{b^2}{c^2}y_2+\frac{c^2}{a^2}y_3}{\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}}\right)$$

erratum: Wrong in 2.1.3 - Apply Lemma 1 -

Ansatz 1: The algebraic expression

Getting an algebraic expression

Use x_3 and y_3 as variables and pluck them into the second-order curve $ax_3^2 + bx_3y_3 + cy_3^2 + dx_3 + ey_3 + f = 0$:

$$x_3 = \frac{(l+u+v)x - lx_1 - ux_2}{v},$$

$$y_3 = \frac{(l+u+v)y - ly_1 - uy_2}{v}.$$

proof idea: Solve $(x, y) = \left(\frac{lx_1+ux_2+vx_3}{l+u+v}, \frac{ly_1+uy_2+vy_3}{l+u+v}\right)$ for x_3, y_3.

erratum: v missing in 2.2.1

Question 3. The Circumcenters' locus

Circumcentres' locus

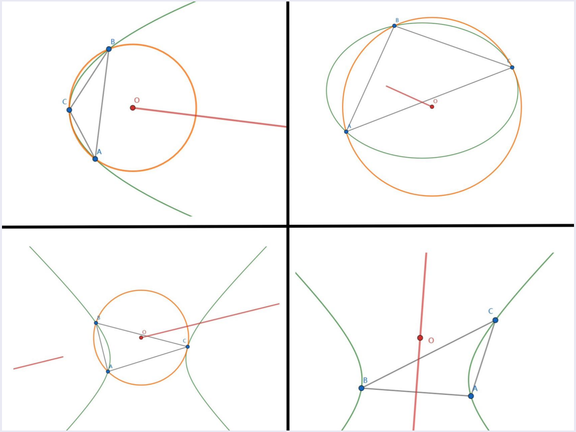
Locus is on the straight line.

$$y = -\frac{x_1 - x_2}{y_1 - y_2}x + \frac{x_1^2 + y_1^2 - x_2^2 - y_2^2}{2y_1 - 2y_2}$$

proof idea: Circumcentre is on perpendicular bisector of A and B.
missing argument: Line segment for ellipse; Ray for parabola; (intercepted) line for hyperbola
proof idea: Circle through A and B such that its tangent to the curve defines the range.

Question 3. The Circumcenters' locus

Different cases



Question 4. The Centre of Gravitys' locus

Centre of Gravitys' locus

Has the same shape as the initial second-order curve:

$$a(3x - x_1 - x_2)^2 + b(3x - x_1 - x_2)(3y - y_1 - y_2) +$$
$$c(3y - y_1 - y_2)^2 + d(3x - x_1 - x_2) + e(3y - y_1 - y_2) + f = 0$$

proof idea: Apply G on $ax^2 + bxy + cy^2 + dx + ey + f = 0$

All the other loci

Algebraic expression

Applying this method to the other three situations yields very lengthy formulas.

erratum: The other formulas have to be fixed because of v.

Ansatz 2: The Parametrization

Parametrising the Ellipse

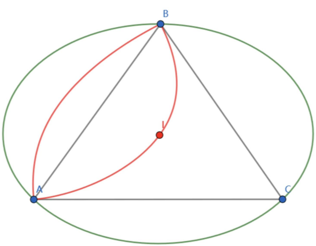
$$\begin{cases} x = \frac{\sqrt{-f}}{\sqrt{a}} \sin t \\ y = \frac{\sqrt{-f}}{\sqrt{c}} \cos t \end{cases}$$

Remark Use the reduced form of ellipse $ax^2 + cy^2 = -f$

Question 1. The Incentres' locus

Ellipse

Lengthy description, but works

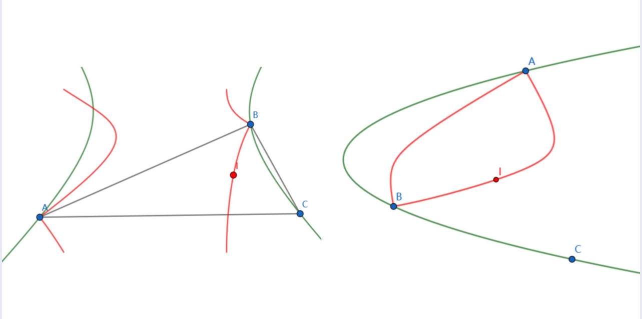


$$
\begin{cases}
x_I = (((\frac{\sqrt{\square}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{\square}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))^2+(-\frac{\sqrt{\square}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+ \\
\frac{\sqrt{\square}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))^2)^{\frac{1}{2}}(\frac{\sqrt{\square}}{\sqrt{a}}sint_1\cos\theta+\frac{\sqrt{\square}}{\sqrt{a}}cost_1\sin\theta-v\cos\theta-w\sin\theta) \\
+((\frac{\sqrt{\square}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{\square}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))^2+(-\frac{\sqrt{\square}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+ \\
\frac{\sqrt{\square}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))^2)^{\frac{1}{2}}(\frac{\sqrt{\square}}{\sqrt{a}}sint_2\cos\theta+\frac{\sqrt{\square}}{\sqrt{a}}cost_2\sin\theta-v\cos\theta-w\sin\theta) \\
+((\frac{\sqrt{\square}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+\frac{\sqrt{\square}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))^2+(-\frac{\sqrt{\square}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+ \\
\frac{\sqrt{\square}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1))^2)^{\frac{1}{2}}(\frac{\sqrt{\square}}{\sqrt{a}}sint_3\cos\theta+\frac{\sqrt{\square}}{\sqrt{a}}cost_3\sin\theta-v\cos\theta-w\sin\theta))/ \\
(((\frac{\sqrt{\square}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{\square}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))^2+(-\frac{\sqrt{\square}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+ \\
\frac{\sqrt{\square}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))^2)^{\frac{1}{2}}+((\frac{\sqrt{\square}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{\square}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))^2+ \\
(-\frac{\sqrt{\square}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{\square}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))^2)^{\frac{1}{2}}+((\frac{\sqrt{\square}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+ \\
\frac{\sqrt{\square}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))^2+(-\frac{\sqrt{\square}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+\frac{\sqrt{\square}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1))^2)^{\frac{1}{2}}) \\
\\
y_I = (((\frac{\sqrt{\square}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{\square}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))^2+(-\frac{\sqrt{\square}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+ \\
\frac{\sqrt{\square}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))^2)^{\frac{1}{2}}(-\frac{\sqrt{\square}}{\sqrt{c}}sint_1\sin\theta+\frac{\sqrt{\square}}{\sqrt{c}}cost_1\cos\theta+v\sin\theta-w\cos\theta) \\
+((\frac{\sqrt{\square}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{\square}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))^2+(-\frac{\sqrt{\square}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+ \\
\frac{\sqrt{\square}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))^2)^{\frac{1}{2}}(-\frac{\sqrt{\square}}{\sqrt{c}}sint_2\sin\theta+\frac{\sqrt{\square}}{\sqrt{c}}cost_2\cos\theta+v\sin\theta-w\cos\theta) \\
+((\frac{\sqrt{\square}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+\frac{\sqrt{\square}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))^2+(-\frac{\sqrt{\square}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+ \\
\frac{\sqrt{\square}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1))^2)^{\frac{1}{2}}(-\frac{\sqrt{\square}}{\sqrt{c}}sint_1\sin\theta+\frac{\sqrt{\square}}{\sqrt{c}}cost_1\cos\theta+v\sin\theta-w\cos\theta))/ \\
(((\frac{\sqrt{\square}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{\square}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))^2+(-\frac{\sqrt{\square}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+ \\
\frac{\sqrt{\square}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))^2)^{\frac{1}{2}}+((\frac{\sqrt{\square}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{\square}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))^2+ \\
(-\frac{\sqrt{\square}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{\square}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))^2)^{\frac{1}{2}}+((\frac{\sqrt{\square}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+ \\
\frac{\sqrt{\square}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))^2+(-\frac{\sqrt{\square}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+\frac{\sqrt{\square}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1))^2)^{\frac{1}{2}})
\end{cases}
$$

Properties: continuous, bounded, contains A and B

Question 1. The Incentres' locus

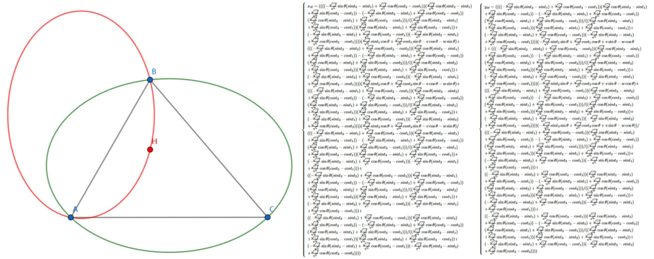
Hyperbola and Parabola



Properties: continuous (2 parts for the 2 parts of hyperbola), contains A and B, bounded for parabola

Question 2. The Orthocenters' locus

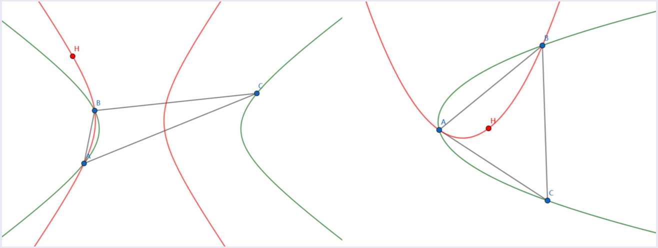
The Ellipse



Properties: Continous contains A and B, bounded

Question 2. The Orthocenters' locus

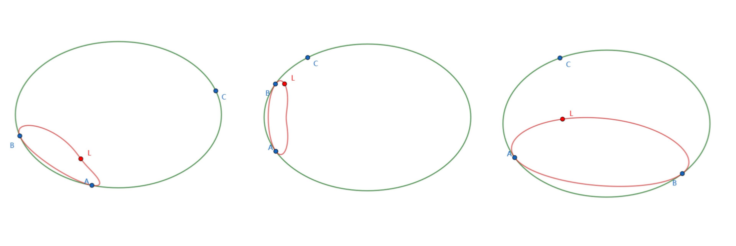
Hyperbola and Parabola



Properties: Continous (per hyperbola-branch), contains A and B, not bounded

Question 5. The Intersection of Symmedians' locus

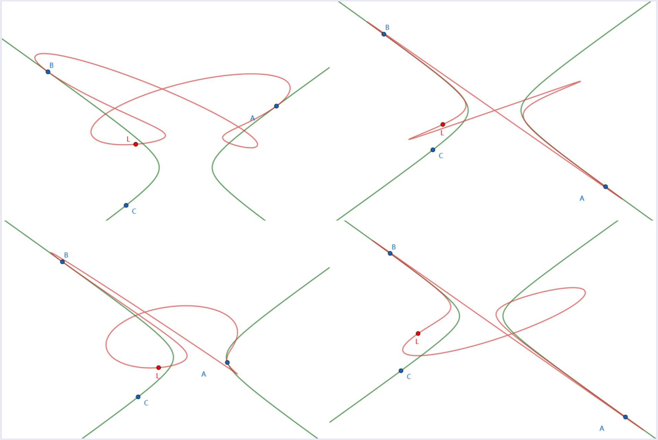
The Ellipse



Properties: Continous, contains A and B, bounded

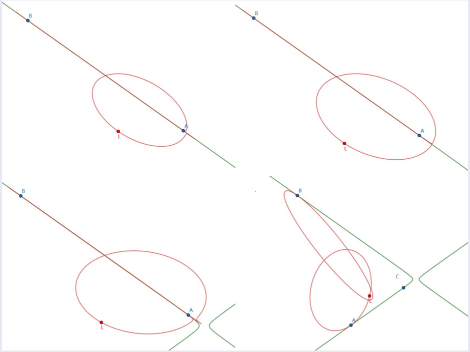
Question 5. The Intersection of Symmedians' locus

Hyperbola



Question 5. The Intersection of Symmedians' locus

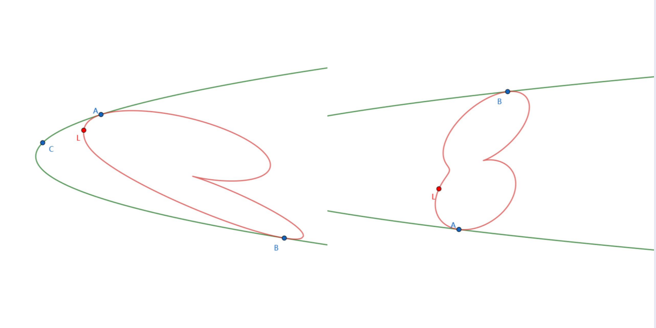
Hyperbola



Properties: Continous (per branch), bounded, Contains A,B

Question 5. The Intersection of Symmedians' locus

Parabola



Properties: Continous, Contains A,B, bounded

Summary

Summary

1. Getting the right coordinates with Lemma 1
2. Algebraic expression by making x_3 and y_3 the new variables
3. Parametrizing the curves