A Unified Variational Framework for Planar Elastica under Magnetic and Gravitational Loads

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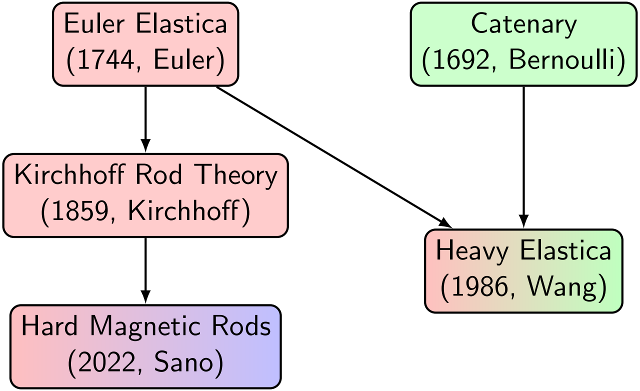
Outline

- Motivation: from Classical Elastica to Magnetic Rods
- The Main Difficulty
- A unified variational framework: from nested integrals to cumulative field functions
- Application I: Hard magnetic rods
- Application II: Gravity, heavy elastica and the catenary limit
- Conclusions and outlook

From Classical Elastica to Magnetic Rods



Euler Elastica (1744, Euler)

Catenary (1692, Bernoulli)

Kirchhoff Rod Theory (1859, Kirchhoff)

Heavy Elastica (1986, Wang)

Hard Magnetic Rods (2022, Sano)

🟥 Classical elasticity and rod mechanics

🟩 Gravity and distributed weight effects

🟦 Magnetic interactions and magnetic loading

The Main Difficulty

In variational formulations, distributed loads often introduce nested integrals into the energy functional. Also the nested integrals leads to integro-differential equations.

Gravity

$$E_g = g \int_0^L y(s)\, ds = g \int_0^L \left(\int_0^s \sin\theta(\sigma)\, d\sigma \right) ds$$

Hard Magnetic Rods (Sano, 2022)

$$EI\theta''(s) - \frac{AB^r b}{\mu_0} \left(y(s) \sin\theta(s) - \cos\theta(s) \int_s^L \cos\theta(s')\, ds' \right) = 0.$$

Different physical mechanisms $\Rightarrow$ Nested integral structures
$\Rightarrow$ Integro-differential equations

A Unified Variational Framework - Nested Integral

Abstracting from specific physical models, we consider the generic nested integral

$$I = \int_0^L \int_0^{s_1} g_1(s_1) g_2(s_2)\, ds_2\, ds_1.$$

This structure appears naturally in distributed-load variational problems.

A Unified Variational Framework - Nested Integral

By applying the **Fubini's Theorem**

$$I := \int_0^L \int_0^{s_1} g_1(s_1) g_2(s_2) \mathrm{d}s_2 \mathrm{d}s_1 = \int_0^L \int_{s_2}^L g_1(s_1) g_2(s_2) \mathrm{d}s_1 \mathrm{d}s_2$$

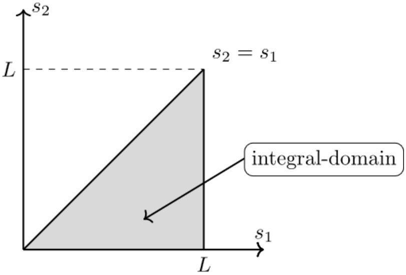


Figure 1: Triangular integration domain in the (s_1, s_2) plane (Mao et al. [1]).

A Unified Variational Framework - Cumulative function

We define here a **Cumulative function**

$$G(s_2) := \int_{s_2}^{L} g_1(\xi)\,\mathrm{d}\xi$$

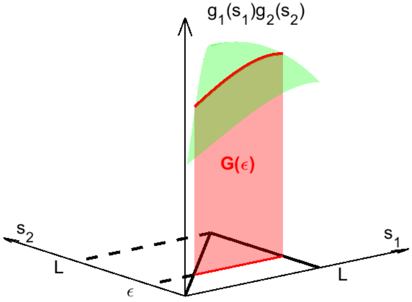


Figure 2: Geometric interpretation of the cumulative field function $G(s)$ (Mao et al. [1]).

A Unified Variational Framework - Summary

$$I = \int_0^L \int_0^{s_1} g_1(s_1) g_2(s_2)\, ds_2\, ds_1 = \int_0^L G(s_2)\, g_2(s_2)\, ds_2 = \int_0^L G(s)\, g_2(s)\, ds,$$

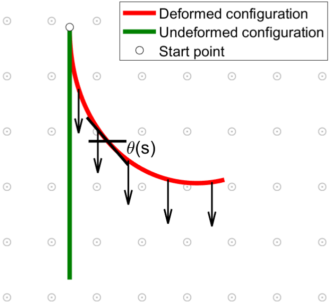
where

$$G(s) = \int_s^L g_1(\xi)\, d\xi.$$

Nested Integral $\Longrightarrow$ Single Integral

Applications of the Framework

We consider planar elastic rods subjected to different distributed external fields due to Mao et al. [1].



- Bending stiffness EI
- Residual magnetization B^r
- Applied magnetic field $\mathbf{B}^a$
- Gravitational field g

Variational Formulation

According to the variational formulation proposed by Sano et al. (2022), the total energy is

$$E_{\mathrm{total}} = \int_0^L \frac{EI}{2}(\theta')^2 \mathrm{d}s - \frac{A}{\mu_0}\int_0^L (\mathbf{F}\mathbf{B}^r)\cdot \mathbf{B}^a \mathrm{d}s$$

For a planar inextensible rod, the tangent vector is $\mathbf{t}(s) = (\cos\theta(s), \sin\theta(s))$, and the residual magnetization is assumed to align with the rod centerline,

$$\mathbf{F}\mathbf{B}^r = B^r \mathbf{t}(s).$$

Therefore, the magnetic energy reduces to

$$E_{\mathrm{mag}} = -\frac{AB^r}{\mu_0}\int_0^L \Big(B_x^a(s)\cos\theta(s) + B_y^a(s)\sin\theta(s)\Big)\mathrm{d}s. \qquad (1)$$

Variational Formulation

Express θ via

$$\theta(s) = \theta(0) + \int_0^s \theta'(\tau) \mathrm{d}\tau$$

which allows us to expand $\sin\theta(s)$ and $\cos\theta(s)$ as

$$\sin\theta(s) = \sin\theta(0) + \int_0^s \theta'(\tau)\cos\theta(\tau)\mathrm{d}\tau$$
$$\cos\theta(s) = \cos\theta(0) - \int_0^s \theta'(\tau)\sin\theta(\tau)\mathrm{d}\tau.$$

Applying to (1) yields

$$E_{\text{mag}} = \frac{AB^r}{\mu_0}\int_0^L \theta'(\tau)\left(\sin\theta(\tau)\Big(\int_\tau^L B_x^a(s)\,\mathrm{d}s\Big) - \cos\theta(\tau)\Big(\int_\tau^L B_y^a(s)\,\mathrm{d}s\Big)\right)\mathrm{d}\tau + \mathrm{C}_1.$$

Variational Formulation

We define cumulative field functions:

$$K_x^a(\tau) = \int_\tau^L B_x^a(s)\mathrm{d}s, \qquad K_y^a(\tau) = \int_\tau^L B_y^a(s)\mathrm{d}s.$$

The magnetic energy is described by

$$\begin{aligned} E_{\text{mag}} &= \frac{AB^r}{\mu_0} \int_0^L \theta'(\tau)\left(\sin\theta(\tau)K_x^a(\tau) - \cos\theta(\tau)K_y^a(\tau)\right)\mathrm{d}\tau + \mathrm{C}_1 \\ &= \frac{AB^r}{\mu_0} \int_0^L \theta'(s)\left(\sin\theta(s)K_x^a(s) - \cos\theta(s)K_y^a(s)\right)\mathrm{d}\tau + \mathrm{C}_1 \end{aligned} \tag{2}$$

This yields the Euler-Lagrange equation

$$EI\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + \frac{AB^r}{\mu_0}\left(\sin\theta(s)K_x^a(s) - \cos\theta(s)K_y^a(s)\right) = 0. \tag{3}$$

Recovery of Sano's Equation

For a constant gradient magnetic field $\mathbf{B}^a = b\,y\,\mathbf{e}_y$,
we have

$$B_x^a = 0, \qquad B_y^a = b\,y(s), \qquad K_x^a(s) = 0, \qquad K_y^a(s) = b\int_s^L y(s')\,ds'.$$

Substituting into the general equation

$$EI\theta'' + \frac{AB^r}{\mu_0}\left(\sin\theta K_x^a - \cos\theta K_y^a\right) = 0,$$

gives

$$EI\theta'' - \frac{AB^r b}{\mu_0}\cos\theta(s)\int_s^L y(s')\,ds' = 0.$$

Applying Fubini's theorem once more yields

$$EI\theta'' - \frac{AB^r b}{\mu_0}\left(y(s)\sin\theta(s) - \cos\theta(s)\int_s^L \cos\theta(s')\,ds'\right) = 0.$$

Numerical Simulation

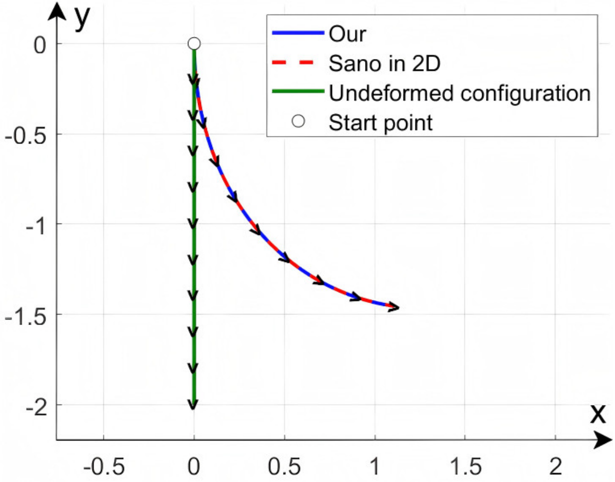


Figure 3: Comparison of the undeformed configuration with constant magnetic field with field lines perpendicular to the paper plane due to Mao et al. [1]. In blue is our solution and in red the solution of Sano et al.[8]

Extension to the $X-Z$ Plane with Gravity

To include gravitational effects, we extend the formulation from the $X-Y$ plane to the $X-Z$ plane,

$$x'(s) = \cos\theta(s), \qquad z'(s) = \sin\theta(s).$$

The gravitational potential energy is

$$E_{\mathrm{grav}} = \rho A g \int_0^L z(s)\, ds.$$

Applying the same integral-reduction framework yields

$$E_{\mathrm{grav}} = \rho A g \int_0^L (L-s)\sin\theta(s)\, ds.$$

Combining bending, magnetic and gravitational contributions gives

$$EI\,\theta''(s) - \frac{AB^r}{\mu_0}\Big(-\sin\theta(s)\,K_x^a(s) + \cos\theta(s)\,K_z^a(s)\Big) + \rho A g (L-s)\cos\theta(s) = 0.$$

Numerical Simulation

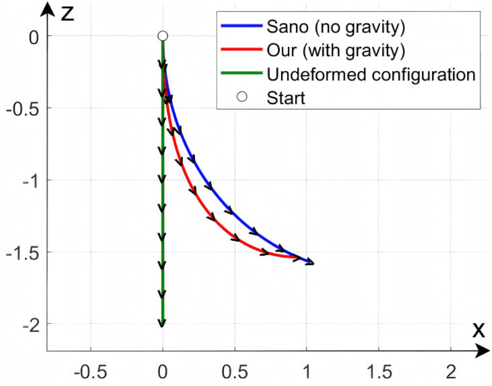


Figure 4: Comparison of the undeformed configuration with constant magnetic field. In addition to figure 3 we have gravitational force in $-z$-direction (red) (Mao et al. [1]). The solution of Sano et al. [8] is also represented (blue) to point out the difference to the nongravitational case.

Heavy Elastica and the Catenary Limit

Setting the magnetic field to zero and introducing the Lagrange multiplier F_1, the governing equation becomes

$$EI\,\theta''(s) = \lambda g(L-s)\cos\theta(s) - F_1\sin\theta(s).$$

This is precisely the classical heavy elastica equation.

Introducing

$$H = \frac{-F_1}{EI}, \qquad V = \frac{-\lambda g L}{EI}, \qquad B = \frac{\lambda g}{EI},$$

yields Wang's form (1986)

$$\theta''(s) = H\sin\theta(s) - (V+Bs)\cos\theta(s).$$

Moreover,

$$\lambda g = 0 \quad\Longrightarrow\quad \text{Euler Elastica}, \quad EI \to 0 \quad\Longrightarrow\quad \text{Classical Catenary}.$$

Numerical Simulation

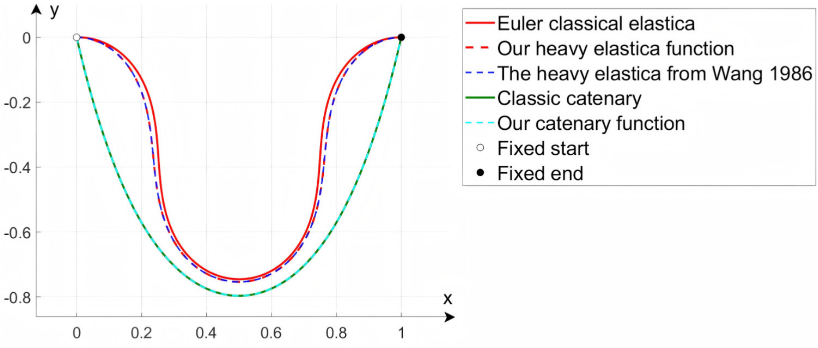


Figure 5: Comparison of the classical catenary and the classical elastica solutions with the heavy elastica solutions (Mao et al. [1]). Our solution matches the solution of Wang [9].

Conclusions and Outlook

1. We proposed a new viewpoint for variational problems involving distributed loads.
2. Nested integral structures can be reformulated using cumulative field functions.
3. This converts nested integrals into compact single-integral energy representations.
4. The framework naturally recovers several classical models:
   - Hard magnetic rods (Sano et al.)
   - Heavy elastica (Wang)
   - The Euler elastica and catenary as limiting cases
5. Potential applications include magnetically actuated soft robots and other field-responsive elastic structures.

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Thank you for your attention.

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