Problem 9. Triangles on Curves.

Team: Germany (A)

Abstract

Given a second order algebraic curve in $\mathbb{R}^2$. Whenever we fix three points on that curve they will form a triangle which come naturally with specific points like the incentres, the intercentres, the orthocentres or the intersection points of the medians or the symmedians. In this paper we describe the occuring loci of these points if one fixes two of the edges A and B of the triangle and moves the third one C freely along the given algebraic curve. The final question is to find a general algebraic expression for the loci. It turns out that these can have rather lengthy descriptions.
At first, we give a general expression for the coordinates of the special points in the triangle. After that, we use it to define the expressions for the loci. We were then able to gain further representations for a variety of different loci by evaluating affine transformations and extensively working with parametric representations of the objects at hand.

Finally, we were able to solve all given problems, but did not yet consider additional fields of study.

Question Result	Letter
1	fully solved
2	partially solved but without a compact formula for the curve
3	fully solved
4	fully solved
5	partially solved but without a compact formula for the curve
6	basic ideas

Contents

1 Introduction and Overview

In this question, we need to analyse a second-order curve. There are three different situations for this second-order curve. The second-order curve can be

- a parabola
- an ellipse (which includes the circle)
- a hyperbola.

Before we start to analyse these three different situations, we can first get the algebraic expression of the coordinates of various special points (namely incentre, othocentre, circumcentres, point of intersection of median and point of intersection of symmedians) within a triangle.

2 Derivation of the main result

2.1 The coordinates of the special points in triangle

2.1.1 Lemma 1: Description of inner points

We mark the three vertices of the triangle as $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$. If points F, D, E divide AB, BC, AC internally or externally, then we have $\frac{AF}{FB} = \lambda_1$, $\frac{BD}{DC} = \lambda_2$, $\frac{CE}{EA} = \lambda_3$, and BE and AD intersect at point B' ; AD and CF intersect at point A' ; CF and BE intersect at C'. The coordinates of these points $A'(x_1', y_1'), B'(x_2', y_2'), C'(x_3', y_3')$ are given by

$$x_i' = \frac{x_i + \lambda_i x_{i+1} + \lambda_i \lambda_{i+1} x_{i+2}}{1 + \lambda_i + \lambda_i \lambda_{i+1}}, y_i' = \frac{y_i + \lambda_i y_{i+1} + \lambda_i \lambda_{i+1} y_{i+2}}{1 + \lambda_i + \lambda_i \lambda_{i+1}}.$$

Let $i + k = m - 3 (i = 1, 2, 3; k = 1, 2)$, when $i + k = m \geq 4$

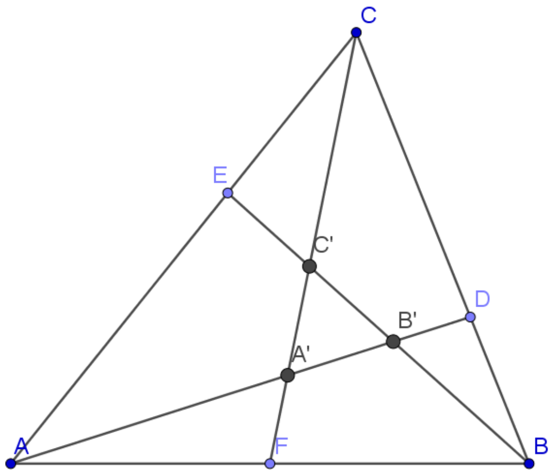

Figure 1: Triangle ΔABC

Proof. For triangle ΔBEC, we can use *Menelaus' theorem*, then we have $\frac{\overline{AE}}{\overline{AC}} \cdot \frac{\overline{CD}}{\overline{DB}} \cdot \frac{\overline{BB'}}{\overline{B'E}} = 1$, namely

$$\frac{\overline{BB'}}{\overline{B'E}} = (1 + \lambda_3)\lambda_2.$$

In the same way we get

$$\frac{\overline{CC'}}{\overline{C'F}} = (1 + \lambda_1)\lambda_3, \frac{\overline{AA'}}{\overline{A'D}} = (1 + \lambda_2)\lambda_1.$$

We suppose $D(d_1,d_2),E(e_1,e_2),F(f_1,f_2)$, and then we can use definite Formula proportion and division point for Line Segments. Therefore

$$d_1=\frac{x_2+\lambda_2 x_3}{1+\lambda_2},e_1=\frac{x_3+\lambda_3 x_1}{1+\lambda_3},f_1=\frac{x_1+\lambda_1 x_2}{1+\lambda_1}$$

Then we have

$$x_1'=\frac{x_1+\lambda_1 x_2+\lambda_3\lambda_1 x_2}{1+\lambda_1+\lambda_1\lambda_2}$$

$$x_2'=\frac{x_2+\lambda_2 x_3+\lambda_2\lambda_3 x_1}{1+\lambda_2+\lambda_2\lambda_3}$$

$$x_3'=\frac{x_3+\lambda_3 x_1+\lambda_1\lambda_2 x_3}{1+\lambda_3+\lambda_3\lambda_1}$$

The same way, we can get the expression of y_1',y_2',y_3' □

2.1.2 List of the desired coordinates within a triangle

So according to Lemma 1, we can choose the correct λ_i to get the desired coordiantes.

1. If $\lambda_1=\frac{n}{m},\lambda_2=\frac{q}{n}\lambda_3=\frac{m}{q}$ and $\overline{BC}=m,\overline{AC}=n,\overline{AB}=q$, the coordinate of **incentre** I is
$$I(\frac{mx_1+nx_2+qx_3}{m+n+q},\frac{my_1+ny_2+qy_3}{m+n+q})$$

2. If $\lambda_1=\frac{\cot A}{\cot B},\lambda_2=\frac{\cot B}{\cot C},\lambda_3=\frac{\cot C}{\cot A}$, the coordinate of the **orthocentre** H is
$$H(\frac{x_1\tan A+x_2\tan B+x_3\tan C}{\tan A+\tan B+\tan C},\frac{y_1\tan A+y_2\tan B+y_3\tan C}{\tan A+\tan B+\tan C})$$

3. Since the **circumcentre** of ΔABC is the orthocentre of ΔDEF, the coordinates of the circumcentre of ΔABC are given by the term
$$O(\frac{x_1(\tan B+\tan C)+x_2(\tan C+\tan A)+x_3(\tan A+\tan B)}{2(\tan A+\tan B+\tan C)},$$
$$\frac{y_1(\tan B+\tan C)+y_2(\tan C+\tan A)+y_3(\tan A+\tan B)}{2(\tan A+\tan B+\tan C}))$$
Let us remark that we need to remove the situation that ΔABC is a right triangle. Consequently, we will treat this case specifically.

4. when $\lambda_1=\lambda_2=\lambda_3=1$,the centre of gravity G is
$$G(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3})$$

2.1.3 Coordinate of intersection of symmedians

It is time for us to talk about the coordinate of intersection of symmedians which we will denote by L.this is more intriguing than the previous cases, we decided to place this in a separate subsection.
As we all know the symmedians are three special lines in a triangle that are defined via the reflection of the bisector of the corresponding angle on the three median lines. According to the definition, it is symmetrical with the median, and the axis of symmetry is the angular bisector. Its construction is shown in figure 2. So we can know that $\angle B'A'L = \angle C'A'E$. We

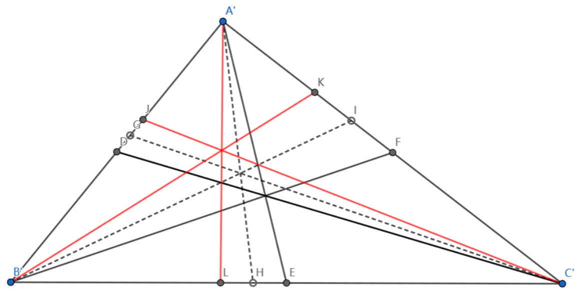

Figure 2: The Intersection L of the three symmedians

will now evolve the coordinates of L in the following way.
We stick to our notation and consider a point D between B and C and instead of Lemma 1, we now want to compare the areas of the resulting inner triangles just like in figure 3. By this, we can identify the coordinate of the symmedian intersection point L. Now we compare

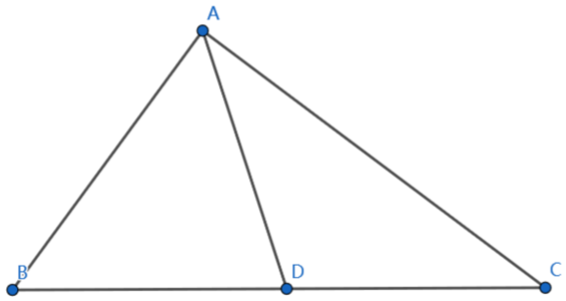

Figure 3: Triangle ΔABC with D on BC

the areas of the two triangles $S_{\Delta ABD}$ and $S_{\Delta ADC}$ and find

$$S_{\Delta ABD} = \frac{1}{2}|\overline{AB}| \cdot |\overline{AD}| \sin\angle BAD = \frac{1}{2}|\overline{BD}| \cdot |\overline{AD}| \sin\angle ADB$$

$$S_{\Delta ADC} = \frac{1}{2}|\overline{AC}| \cdot |\overline{AD}| \sin\angle DAC = \frac{1}{2}|\overline{CD}| \cdot |\overline{AD}| \sin\angle ADC$$

So we have

$$\overline{AB} \sin\angle BAD = \overline{BD} sin\angle ADB$$

$$\overline{AC} \sin\angle DAC = \overline{CD} \sin\angle ADC$$

As we all know that $\sin\angle ADB = \sin\angle ADC$,so we have

$$\frac{\overline{AB} \sin\angle BAD}{\overline{AC} sin\angle DAC} = \frac{\overline{BD}}{\overline{CD}} \tag{1}$$

Use of the formula We can use (1) in the triangle in the Figure 2.

$$\frac{B'L}{LC'} = \frac{A'B'}{A'C'}\frac{\sin\angle B'A'L}{\sin\angle LA'C'}$$

$$\frac{BE}{EC} = \frac{A'B'}{A'C'}\frac{\sin\angle B'A'E}{\sin\angle EA'C'}$$

It is clear that $\sin\angle B'A'E = \sin\angle LA'C', sin\angle B'A'L = \sin\angle EA'C'$ so that we have

$$\frac{B'L}{C'L} = (\frac{A'B'}{A'C'})^2$$

So according to the Lemma 1,we have coordinate intersection of symmedians

$$L(\frac{x_1 + x_2 + x_3 + (\frac{a^2}{c^2} + \frac{a^2}{b^2})x_1 + (\frac{b^2}{a^2} + \frac{b^2}{c^2})x_2 + (\frac{c^2}{a^2} + \frac{c^2}{b^2})x_3}{a^2 + b^2 + c^2},$$

$$\frac{y_1 + y_2 + y_3 + (\frac{a^2}{c^2} + \frac{a^2}{b^2})y_1 + (\frac{b^2}{a^2} + \frac{b^2}{c^2})y_2 + (\frac{c^2}{a^2} + \frac{c^2}{b^2})y_3}{a^2 + b^2 + c^2})$$

*a, b and c are the lengths of the three sides of the triangle.

2.2 The way to describe the shape of the locus

2.2.1 Some preliminary analysis

We are interested in the locus of a point that occurs when we fix A and B but move the point C of the triangle along the given quadratic curve. In the following we rename the three points A, B, C to P_1, P_2, P_3 and search for a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that will describe the shape of the locus. This means: Given the two fixed points P_1 and P_2 of the triangle on the given algebraic curve, we are looking for the function f to describe the locus.

$$(P_1, P_2) \to f \quad \text{or} \quad (P_1, P_2) \to f(x, y)$$

Now we notice that all the coordinate can be written like the following type.

$$(\frac{lx_1 + ux_2 + vx_3}{l + u + v}, \frac{ly_1 + vy_2 + uy_3}{l + u + v})$$

For all the $(x_1, y_1), (x_2, y_2), (x_3, y_3)$: They must be fit the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

If we want to find the relationship or mapping between

$$\frac{lx_1 + ux_2 + vx_3}{l + u + v} \text{ and } \frac{ly_1 + uy_2 + vy_3}{l + u + v}$$

we can research the algebraic structure of these two algebraic expressions. It is very clear that the algebraic structure is made up with some small algebraic expressions, so the mapping φ

$$\varphi : \frac{lx_1 + ux_2 + vx_3}{m + n + q} \to \frac{ly_1 + uy_2 + vy_3}{l + u + v}$$

has the same character as the mapping ν which is $x_1 \to y_1$, $x_2 \to y_2$ or $x_3 \to y_3$: ν is a kind of mapping that makes another mapping $\phi : (x, y) \to f(x, y)$ tenable,so mapping φ also can make the mapping $\phi : (x, y) \to f(x, y)$ tenable, so φ is also a second-order curve type mapping. Anyway, we can construct a mapping that satisfies the conditions, so the locus can be expressed.

No matter how it is, we can make an equation with $\frac{mx_1+nx_2+qx_3}{m+n+q}$ and $\frac{my_1+ny_2+qy_3}{m+n+q}$ from $ax^2+bxy+cy^2+dx+ey+f=0$ By definition the points of the triangle (x_1,y_1),(x_2,y_2) and (x_3,y_3) satisfy the equation $ax^2+bxy+cy^2+dx+ey+f=0$. This means that we have

$$ax_1^2+bx_1y_1+cy_1^2+dx_1+ey_1+f=0$$

$$ax_2^2+bx_2y_2+cy_2^2+dx_2+ey_2+f=0$$

$$ax_3^2+bx_3y_3+cy_3^2+dx_3+ey_3+f=0$$

As we all know that (x_3,y_3) doesn't influence the locus, but it fits a second-order curve, so we can write the following term to make a function to make

$$(\frac{lx_1+ux_2+vx_3}{l+u+v},\frac{ly_1+uy_2+vy_3}{l+u+v})$$

work.
We can use the following things to replace formal the x and y into

$$(l+u+v)x-lx_1-ux_2,(l+u+v)y-ly_1-uy_2.$$

Then we arrive at

$$a((l+u+v)x-lx_1-ux_2)^2+b((l+u+v)x-lx_1-ux_2)((l+u+v)y-ly_1-uy_2)$$

$$+c((l+u+v)y-ly_1-uy_2)^2+d((l+u+v)x-lx_1-ux_2)+e((l+u+v)y-ly_1-uy_2)+f=0.$$

2.2.2 The Incentre's locus

When we talk about the Incentre's, we can just understand the l,u,v as m,n,q, but a,b,c have something to do with the length of the triangle,that means,a,b,c have something to do with the (x_1,y_1),(x_2,y_2) and (x_3,y_3), so it maybe influence the shape of the locus,so we need more analyse

2.2.3 The Orthocentre's locus

When we talk about the orthocentre H, we can just understand the algebraic expression as the same one, namely we can understand l as $\tan A$,u as $\tan B$,v as $\tan C$,and $tanA$, $tanB$ and $tanC$ has something to do with the (x_1,y_1),x_2,y_2 and (x_3,y_3), so it maybe influence the shape of the locus,so we need more analyse

2.2.4 The Circumcentre's locus

When we talk about the Circumcentre O, we can just understand the algebraic expression as the same one, namely we can understand l as $\tan B+\tan C$,u as $\tan C+\tan A$,v as $\tan A+\tan B$,but if we think more about it, the circumcentre must be a line,so we need use another way (the Geometrical method) to prove it. As we all know that Point A,B are socalled fixed, so that it can fix the perpendicular bisector of AB and according to the definition of the circumcentre, the circumcentre is on the perpendicular bisector of AB, so circumcentre's locus is a line

2.2.5 Intersection of medians' locus

When we talk about the intersection of medians' locus G, we can just understand the algebraic expression as the same one, namely we can understand m,n,q as 1,so it is very clear that the locus of intersection of medians' locus is a sencond-order curve.

2.2.6 The locus of the Intersection of symmedians

When we talk about the symmedians locus,we can also understand l,u and v as $\frac{a^2}{a^2}+\frac{a^2}{b^2}+\frac{a^2}{c^2}$,$\frac{b^2}{a^2}+\frac{b^2}{b^2}+\frac{b^2}{c^2}$ and $\frac{c^2}{a^2}+\frac{c^2}{b^2}+\frac{c^2}{c^2}$, but $\frac{a^2}{a^2}+\frac{a^2}{b^2}+\frac{a^2}{c^2}$,$\frac{b^2}{a^2}+\frac{b^2}{b^2}+\frac{b^2}{c^2}$ and $\frac{c^2}{a^2}+\frac{c^2}{b^2}+\frac{c^2}{c^2}$ have something to do with the length of the triangle,that means,they have something to do with the (x_1, y_1),x_2, y_2 and (x_3, y_3), so it maybe influence the shape of the locus,so we need more analyse

2.3 The algebraic expression of the loci

After we find out that we can use mapping and so on to descreibe the shape of locus, we can now try to get the algebraic expression for each locus.

2.3.1 Algebraic description of the incentre's locus

It is time for us to talk about the algebraic exppression for incentre's locus.That means, we need to deal with coordinate of the incentre, and find the relationship from the coordinte,,but what we need to pay attention with is, there are two different situation, one is C above the AB,the another is C under AB,they are different, also with different algebraic expression.So we now look at the algebraic expression of incentre

$$I(\frac{mx_1+nx_2+qx_3}{m+n+q}, \frac{my_1+ny_2+qy_3}{m+n+q})$$

According to the 2.1.2 we know that they fit the following function

$$a((m+n+q)x-mx_1-nx_2)^2+b((m+n+q)x-mx_1-nx_2)((m+n+q)y-my_1-ny_2)$$

$$+c((m+n+q)y-my_1-ny_2)^2+d((m+n+q)x-mx_1-nx_2)+e((m+n+q)y-my_1-ny_2)+f=0$$

As we all know that m, n, q have something to do with the length of the three sides of the triangle, namely

$$m=\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}$$
$$n=\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}$$
$$q=\sqrt{(x_2-y_1)^2+(y_2-y_1)^2}$$

Then we have

$$a((\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}+\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}+\sqrt{(x_2-y_1)^2+(y_2-y_1)^2})x$$
$$-\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}x_1-\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}x_2)^2+$$
$$b((\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}+\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}+\sqrt{(x_2-y_1)^2+(y_2-y_1)^2})x$$
$$-\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}x_1-\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}x_2)$$
$$((\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}+\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}+\sqrt{(x_2-y_1)^2+(y_2-y_1)^2})y$$
$$-\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}y_1-\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}y_2)$$
$$+c((\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}+\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}+\sqrt{(x_2-y_1)^2+(y_2-y_1)^2})y-$$
$$\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}y_1-\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}y_2)^2$$
$$+d((\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}+\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}+\sqrt{(x_2-y_1)^2+(y_2-y_1)^2})x$$
$$-\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}x_1-\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}x_2)$$
$$+e((\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}+\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}+\sqrt{(x_2-y_1)^2+(y_2-y_1)^2})y$$
$$-\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}y_1-\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}y_2)+f=0$$

2.3.2 Algebraic description of the the othocentre's locus

It is time for us to discuss the algebraic expression for the othocentres' locus H.

$$H(\frac{x_1\tan A+x_2\tan B+x_3\tan C}{\tan A+\tan B+\tan C},\frac{y_1\tan A+y_2\tan B+y_3\tan C}{\tan A+\tan B+\tan C})$$

According to the 2.1.2 we can know that they fit the following function

$$a((\tan A+\tan B+\tan C)x-\tan Ax_1-\tan Bx_2)^2+$$

$$b((\tan A+\tan B+\tan C)x-\tan Ax_1-\tan Bx_2)((\tan A+\tan B+\tan C)y-\tan Ay_1-\tan By_2)+$$

$$c((\tan A+\tan B+\tan C)y-\tan Ay_1-\tan By_2)^2+$$

$$d((\tan A+\tan B+\tan C)x-\tan Ax_1-\tan Bx_2)+e((\tan A+\tan B+\tan C)y-\tan Ay_1-\tan By_2)+f=0$$

Now we can use the slope to express the $\tan A$,$\tan B$ and $\tan C$,

$$k_{AB}=\frac{y_2-y_1}{x_2-x_1}$$

$$k_{BC}=\frac{y_3-y_2}{x_3-x_2}$$

$$k_{CA}=\frac{y_3-y_1}{x_3-x_1}$$

and now we can write the tan value for the angle

$$\tan A=\frac{k_{CA}-k_{AB}}{1+k_{AC}k_{AB}}=\frac{\frac{y_3-y_1}{x_3-x_1}-\frac{y_2-y_1}{x_2-x_1}}{1+\frac{y_3-y_1}{x_3-x_1}\frac{y_2-y_1}{x_2-x_1}}=\frac{(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)}{(x_3-x_1)(x_2-x_1)+(y_3-y_1)(y_2-y_1)}$$

$$\tan B=\frac{k_{BC}-k_{AB}}{1+k_{BC}k_{AB}}=\frac{\frac{y_3-y_2}{x_3-x_2}-\frac{y_2-y_1}{x_2-x_1}}{1+\frac{y_3-y_2}{x_3-x_2}\frac{y_2-y_1}{x_2-x_1}}=\frac{(y_3-y_2)(x_2-x_1)-(y_2-y_1)(x_3-x_2)}{(x_3-x_2)(x_2-x_1)+(y_3-y_2)(y_2-y_1)}$$

$$\tan C=\frac{k_{CA}-k_{BC}}{1+k_{CA}k_{BC}}=\frac{\frac{y_3-y_1}{x_3-x_1}-\frac{y_3-y_2}{x_3-x_2}}{1+\frac{y_3-y_1}{x_3-x_1}\frac{y_3-y_2}{x_3-x_2}}=\frac{(y_3-y_1)(x_3-x_2)-(y_3-y_2)(x_3-x_1)}{(x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_2)}$$

It is very clearly that we have

$$a((\frac{(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)}{(x_3-x_1)(x_2-x_1)+(y_3-y_1)(y_2-y_1)}+\frac{(y_3-y_2)(x_2-x_1)-(y_2-y_1)(x_3-x_2)}{(x_3-x_2)(x_2-x_1)+(y_3-y_2)(y_2-y_1)}+$$

$$\frac{(y_3-y_1)(x_3-x_2)-(y_3-y_2)(x_3-x_1)}{(x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_2)})x-\frac{(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)}{(x_3-x_1)(x_2-x_1)+(y_3-y_1)(y_2-y_1)}x_1-$$

$$\frac{(y_3-y_2)(x_2-x_1)-(y_2-y_1)(x_3-x_2)}{(x_3-x_2)(x_2-x_1)+(y_3-y_2)(y_2-y_1)}x_2)^2+$$

$$b((\frac{(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)}{(x_3-x_1)(x_2-x_1)+(y_3-y_1)(y_2-y_1)}+\frac{(y_3-y_2)(x_2-x_1)-(y_2-y_1)(x_3-x_2)}{(x_3-x_2)(x_2-x_1)+(y_3-y_2)(y_2-y_1)}+$$

$$\frac{(y_3-y_1)(x_3-x_2)-(y_3-y_2)(x_3-x_1)}{(x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_2)})x-\frac{(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)}{(x_3-x_1)(x_2-x_1)+(y_3-y_1)(y_2-y_1)}x_1-$$

$$\frac{(y_3-y_2)(x_2-x_1)-(y_2-y_1)(x_3-x_2)}{(x_3-x_2)(x_2-x_1)+(y_3-y_2)(y_2-y_1)}x_2)((\frac{(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)}{(x_3-x_1)(x_2-x_1)+(y_3-y_1)(y_2-y_1)}+$$

$$\frac{(y_3-y_2)(x_2-x_1)-(y_2-y_1)(x_3-x_2)}{(x_3-x_2)(x_2-x_1)+(y_3-y_2)(y_2-y_1)}+\frac{(y_3-y_1)(x_3-x_2)-(y_3-y_2)(x_3-x_1)}{(x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_2)})y-$$

$$-\frac{(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)}{(x_3-x_1)(x_2-x_1)+(y_3-y_1)(y_2-y_1)}y_1-\frac{(y_3-y_2)(x_2-x_1)-(y_2-y_1)(x_3-x_2)}{(x_3-x_2)(x_2-x_1)+(y_3-y_2)(y_2-y_1)}y_2)+$$
$$c((\frac{(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)}{(x_3-x_1)(x_2-x_1)+(y_3-y_1)(y_2-y_1)}+\frac{(y_3-y_2)(x_2-x_1)-(y_2-y_1)(x_3-x_2)}{(x_3-x_2)(x_2-x_1)+(y_3-y_2)(y_2-y_1)}+$$
$$\frac{(y_3-y_1)(x_3-x_2)-(y_3-y_2)(x_3-x_1)}{(x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_2)})y-\frac{(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)}{(x_3-x_1)(x_2-x_1)+(y_3-y_1)(y_2-y_1)}y_1$$
$$-\frac{(y_3-y_2)(x_2-x_1)-(y_2-y_1)(x_3-x_2)}{(x_3-x_2)(x_2-x_1)+(y_3-y_2)(y_2-y_1)}y_2)^2+$$
$$d((\frac{(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)}{(x_3-x_1)(x_2-x_1)+(y_3-y_1)(y_2-y_1)}+\frac{(y_3-y_2)(x_2-x_1)-(y_2-y_1)(x_3-x_2)}{(x_3-x_2)(x_2-x_1)+(y_3-y_2)(y_2-y_1)}+$$
$$\frac{(y_3-y_1)(x_3-x_2)-(y_3-y_2)(x_3-x_1)}{(x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_2)})x-\frac{(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)}{(x_3-x_1)(x_2-x_1)+(y_3-y_1)(y_2-y_1)}x_1-$$
$$\frac{(y_3-y_2)(x_2-x_1)-(y_2-y_1)(x_3-x_2)}{(x_3-x_2)(x_2-x_1)+(y_3-y_2)(y_2-y_1)}x_2)+$$
$$e((\frac{(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)}{(x_3-x_1)(x_2-x_1)+(y_3-y_1)(y_2-y_1)}+\frac{(y_3-y_2)(x_2-x_1)-(y_2-y_1)(x_3-x_2)}{(x_3-x_2)(x_2-x_1)+(y_3-y_2)(y_2-y_1)}+$$
$$\frac{(y_3-y_1)(x_3-x_2)-(y_3-y_2)(x_3-x_1)}{(x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_2)})y-\frac{(y_3-y_1)(x_2-x_1)-(y_2-y_1)(x_3-x_1)}{(x_3-x_1)(x_2-x_1)+(y_3-y_1)(y_2-y_1)}y_1-$$
$$\frac{(y_3-y_2)(x_2-x_1)-(y_2-y_1)(x_3-x_2)}{(x_3-x_2)(x_2-x_1)+(y_3-y_2)(y_2-y_1)}y_2)+f=0$$

2.3.3 Algebraic expression of the circumcentre's locus

As we all know, the graph for the cirumcentre's locus pass the middle of the line AB and perpendicular to the line AB, so we can get the algebraic expression:

$$y=-\frac{x_1-x_2}{y_1-y_2}x+\frac{x_1^2+y_1^2-x_2^2-y_2^2}{2y_1-2y_2}$$

2.3.4 The algebraic expression of the locus of the intersection of the medians

As we all know, we can use definied proportion and division point to get the algebraic expression of the median's locus. If the point the A, B are fixed, so the middle point of A, B is fixed we named the middle point of $\overline{AB}$ as P. If we mark $\frac{PG}{GC}=\lambda$,we have

$$\lambda=\frac{1}{2}=\frac{PG}{GC}=\frac{x_G-x_P}{x_G-x_C}=\frac{y_G-y_P}{y_G-y_C}$$

When we look at this, we can now describe the coordinate of the intersection of the median's like following

$$x_G=\frac{x_p+\lambda x_3}{1+\lambda}, y_G=\frac{y_p+\lambda y_3}{1+\lambda}$$

Because the (x_p, y_p) are fixed, so that (x_G, y_G) can only be influenced by (x_3, y_3) and therefore (x_3, y_3) is on a second-order curve. Hence, (x_G, y_G) must be on a second-order curve as well.

$$x_G=\frac{2x_p+x_3}{3}, y_G=\frac{2y_p+y_3}{2}$$

It is easy for us to know that the locus (x_G, y_G) is the graph is narrowed by three times the graph of equation $ax^2+bxy+cy^2+dx+ey+f=0$, it is like $a(3x)^2+b(3x)(3y)+c(3y)^2+d(3x)+e(3y)+f=0$ and this graph move in the coorinate system, the position of graph is decided by (x_1, y_1) and (x_2, y_2). We can now the use the 2.2.1

$$a(3x-x_1-x_2)^2+b(3x-x_1-x_2)(3y-y_1-y_2)+$$
$$c(3y-y_1-y_2)^2+d(3x-x_1-x_2)+e(3y-y_1-y_2)+f=0$$

2.3.5 Algebraic description of the locus of the intersection of symmedians

If we want to get the algebraic expression for symmedian's locus, we can just replace l, u and v by $\frac{a^2}{a^2}+\frac{a^2}{b^2}+\frac{a^2}{c^2}, \frac{b^2}{a^2}+\frac{b^2}{b^2}+\frac{b^2}{c^2}$ and $\frac{c^2}{a^2}+\frac{c^2}{b^2}+\frac{c^2}{c^2}$, then know that a, b and c have something to do with the length of the sides. This means

$$a^2=(x_3-x_2)^2+(y_3-y_2)^2$$

$$b^2=(x_3-x_1)^2+(y_3-y_1)^2$$

$$c^2=(x_2-y_1)^2+(y_2-y_1)^2$$

Then we have

$$1+\frac{a^2}{b^2}+\frac{a^2}{c^2}=1+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}$$

$$1+\frac{b^2}{a^2}+\frac{b^2}{c^2}=1+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_2-y_1)^2+(y_2-y_1)^2}$$

$$1+\frac{c^2}{a^2}+\frac{c^2}{b^2}=1+\frac{(x_2-y_1)^2+(y_2-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}+\frac{(x_2-y_1)^2+(y_2-y_1)^2}{(x_3-x_1)^2+(y_3-y_1)^2}$$

According to the 2.2.1, we have the following equations

$$a((3+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}+$$

$$\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_2-y_1)^2+(y_2-y_1)^2}+\frac{(x_2-y_1)^2+(y_2-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}+\frac{(x_2-y_1)^2+(y_2-y_1)^2}{(x_3-x_1)^2+(y_3-y_1)^2})x-$$

$$(1+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2})x_1-(1+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}+$$

$$\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_2-y_1)^2+(y_2-y_1)^2})x_2)^2+b((3+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+$$

$$\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_2-y_1)^2+(y_2-y_1)^2}+\frac{(x_2-y_1)^2+(y_2-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}$$

$$+\frac{(x_2-y_1)^2+(y_2-y_1)^2}{(x_3-x_1)^2+(y_3-y_1)^2})x-(1+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2})x_1$$

$$-(1+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_2-y_1)^2+(y_2-y_1)^2})x_2)$$

$$((3+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}$$

$$+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_2-y_1)^2+(y_2-y_1)^2}+\frac{(x_2-y_1)^2+(y_2-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}+\frac{(x_2-y_1)^2+(y_2-y_1)^2}{(x_3-x_1)^2+(y_3-y_1)^2})y$$

$$-(1+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2})y_1-(1+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}+$$

$$\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_2-y_1)^2+(y_2-y_1)^2})y_2)+c((3+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+$$

$$\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_2-y_1)^2+(y_2-y_1)^2}+\frac{(x_2-y_1)^2+(y_2-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}+\frac{(x_2-y_1)^2+(y_2-y_1)^2}{(x_3-x_1)^2+(y_3-y_1)^2})y$$

$$-(1+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2})y_1-(1+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}$$

$$+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_2-y_1)^2+(y_2-y_1)^2})y_2)^2+d((3+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+$$
$$\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_2-y_1)^2+(y_2-y_1)^2})x-(1+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+$$
$$\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2})x_1-(1+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_2-y_1)^2+(y_2-y_1)^2})x_2)$$
$$+e((3+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}+$$
$$\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_2-y_1)^2+(y_2-y_1)^2})y-(1+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2}+\frac{(x_3-x_2)^2+(y_3-y_2)^2}{(x_3-x_1)^2+(y_3-y_1)^2})y_1$$
$$-(1+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_3-x_2)^2+(y_3-y_2)^2}+\frac{(x_3-x_1)^2+(y_3-y_1)^2}{(x_2-y_1)^2+(y_2-y_1)^2})y_2)+f=0$$

2.3.6 Some discoveries

It is appearent that the way how we describe the function f of the loci is not correct, if we treat x_3 ansd y_3 like normal coefficients. All graphs of the functions f should be second-order-curves. But on the other hand we observe that they are not, if we try to draw the graphs. This means that we need a different way to deal the algebraic expressions.

2.4 A different attempt to describe the algebraic expressions of orthocentre's, incentre's and symmedian's locus

Now we are going to try to use the parametric equation to deal with the function, that means we are now trying the find the parametric equation expression for 2.3.1, 2.3.2 and 2.3.5, but as we all know that for different types of graphs(like ellipse, hyperbola or parabola), we have different kinds of prametric equations

2.4.1 Affine transformation

We now assume a point in $\mathbb{R}^2$ in the old coordinate system xOy and now transform it to new coordinates $x'O'y'$. This means we change the coordinate system from (x,y) to(x',y'). So the origin O in the old system will be moved towards O' in the new system. In the old system it belongs to the coordinates (v,w). Therefore, we have

$$\begin{cases} x' = x\cos\theta + y\sin\theta - p\cos\theta - q\sin\theta \\ y' = -x\sin\theta + y\cos\theta + p\sin\theta - q\cos\theta \end{cases}$$

θ is the angle which the second-order-curve turned.

2.4.2 Incentre's locus

Ellipse For an ellipse $ax^2 + bxy + cy^2 + dx + ey + f = 0$, we can get the algebraic expression for its "original" function as $ax^2 + cy^2 = -f$, so we all can get the prametric equations

$$\begin{cases} x = \frac{\sqrt{-f}}{\sqrt{a}} \\ y = \frac{\sqrt{-f}}{\sqrt{c}} \end{cases}$$

Now we can get the prametric equation for the usual ellipse $ax^2 + bxy + cy^2 + dx + ey + f = 0$. That means

$$\begin{cases} x = \frac{\sqrt{-f}}{\sqrt{a}} \cos\theta + \frac{\sqrt{-f}}{\sqrt{a}} \sin\theta - v\cos\theta - w\sin\theta \\ y = -\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta + \frac{\sqrt{-f}}{\sqrt{c}} \cos\theta + v\sin\theta - w\cos\theta \end{cases}$$

So we have

$$\begin{cases} x_1 = \frac{\sqrt{-f}}{\sqrt{a}} \sin t_1 \cos\theta + \frac{\sqrt{-f}}{\sqrt{a}} \cos t_1 \sin\theta - v\cos\theta - w\sin\theta \\ y_1 = -\frac{\sqrt{-f}}{\sqrt{c}} \sin t_1 \sin\theta + \frac{\sqrt{-f}}{\sqrt{c}} \cos t_1 \cos\theta + v\sin\theta - w\cos\theta \end{cases}$$

$$\begin{cases} x_2 = \frac{\sqrt{-f}}{\sqrt{a}} \sin t_2 \cos\theta + \frac{\sqrt{-f}}{\sqrt{a}} \cos t_2 \sin\theta - v\cos\theta - w\sin\theta \\ y_2 = -\frac{\sqrt{-f}}{\sqrt{c}} \sin t_2 \sin\theta + \frac{\sqrt{-f}}{\sqrt{c}} \cos t_2 \cos\theta + v\sin\theta - w\cos\theta \end{cases}$$

$$\begin{cases} x_3 = \frac{\sqrt{-f}}{\sqrt{a}} \sin t_3 \cos\theta + \frac{\sqrt{-f}}{\sqrt{a}} \cos t_3 \sin\theta - v\cos\theta - w\sin\theta \\ y_3 = -\frac{\sqrt{-f}}{\sqrt{c}} \sin t_1 \sin\theta + \frac{\sqrt{-f}}{\sqrt{c}} \cos t_1 \cos\theta + v\sin\theta - w\cos\theta \end{cases}$$

$$\begin{cases} m = ((\frac{\sqrt{-f}}{\sqrt{a}} \cos\theta(\sin t_3 - \sin t_2) + \frac{\sqrt{-f}}{\sqrt{a}} \sin\theta(\cos t_3 - \cos t_2))^2 + (-\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta(\sin t_3 - \sin t_2) + \\ \frac{\sqrt{-f}}{\sqrt{c}} \cos\theta(\cos t_3 - \cos t_2))^2)^{\frac{1}{2}} \\ n = ((\frac{\sqrt{-f}}{\sqrt{a}} \cos\theta(\sin t_3 - \sin t_1) + \frac{\sqrt{-f}}{\sqrt{a}} \sin\theta(\cos t_3 - \cos t_1))^2 + (-\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta(\sin t_3 - \sin t_1) + \\ \frac{\sqrt{-f}}{\sqrt{c}} \cos\theta(\cos t_3 - \cos t_1))^2)^{\frac{1}{2}} \\ q = ((\frac{\sqrt{-f}}{\sqrt{a}} \cos\theta(\sin t_2 - \sin t_1) + \frac{\sqrt{-f}}{\sqrt{a}} \sin\theta(\cos t_2 - \cos t_1))^2 + (-\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta(\sin t_2 - \sin t_1) + \\ \frac{\sqrt{-f}}{\sqrt{c}} \cos\theta(\cos t_2 - \cos t_1))^2)^{\frac{1}{2}} \end{cases}$$

Then we have

$$\begin{cases} x_I=(((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))^2+(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\\ \frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))^2)^{\frac{1}{2}}(\frac{\sqrt{-f}}{\sqrt{a}}\sin t_1\cos\theta+\frac{\sqrt{-f}}{\sqrt{a}}\cos t_1\sin\theta-v\cos\theta-w\sin\theta)\\ +((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))^2+(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\\ \frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))^2)^{\frac{1}{2}}(\frac{\sqrt{-f}}{\sqrt{a}}\sin t_2\cos\theta+\frac{\sqrt{-f}}{\sqrt{a}}\cos t_2\sin\theta-v\cos\theta-w\sin\theta)\\ +((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))^2+(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+\\ \frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1))^2)^{\frac{1}{2}}(\frac{\sqrt{-f}}{\sqrt{a}}\sin t_3\cos\theta+\frac{\sqrt{-f}}{\sqrt{a}}\cos t_3\sin\theta-v\cos\theta-w\sin\theta))/\\ (((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))^2+(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\\ \frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))^2)^{\frac{1}{2}}+((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))^2+\\ (-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))^2)^{\frac{1}{2}}+((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+\\ \frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))^2+(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1))^2)^{\frac{1}{2}})\\ \\ y_I=(((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))^2+(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\\ \frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))^2)^{\frac{1}{2}}(-\frac{\sqrt{-f}}{\sqrt{c}}\sin t_1\sin\theta+\frac{\sqrt{-f}}{\sqrt{c}}\cos t_1\cos\theta+v\sin\theta-w\cos\theta)\\ +((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))^2+(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\\ \frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))^2)^{\frac{1}{2}}(-\frac{\sqrt{-f}}{\sqrt{c}}\sin t_2\sin\theta+\frac{\sqrt{-f}}{\sqrt{c}}\cos t_2\cos\theta+v\sin\theta-w\cos\theta)\\ +((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))^2+(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+\\ \frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1))^2)^{\frac{1}{2}}(-\frac{\sqrt{-f}}{\sqrt{c}}\sin t_1\sin\theta+\frac{\sqrt{-f}}{\sqrt{c}}\cos t_1\cos\theta+v\sin\theta-w\cos\theta))/\\ (((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))^2+(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\\ \frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))^2)^{\frac{1}{2}}+((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))^2+\\ (-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))^2)^{\frac{1}{2}}+((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+\\ \frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))^2+(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1))^2)^{\frac{1}{2}}) \end{cases}$$

We now can write the parametric equation for Incentre. The reason why we get the parametric equation is we now want to know about the function better, in the nomal way, there are only one changing coefficient,in our mind, we think that one changing coefficient should be changing continuous and linear, but now we can know from the normal function and the parametric function, we know that it the so-called "one coefficient" is actually changing not linear, it is actually a kind of transformation of quadratic type and root type.
From this we can realize that it actually has four different variables, not three different variables.

Hyperbola The same way, we can define the "orginal" function for $ax^2+bxy+cy^2+dx+ey+f=0$ as $ax^2+cy^2=-f$, so that we have

$$\begin{cases} x=\frac{\sqrt{-f}}{\sqrt{a}\cos t}\\ y=\frac{\sqrt{-f}\tan t}{\sqrt{-c}} \end{cases}$$

Then we have

$$\begin{cases} x'=\frac{\sqrt{-f}}{\sqrt{a}\cos t}\cos\theta+\frac{\sqrt{-f}\tan t}{\sqrt{-c}}\sin\theta-p\cos\theta-q\sin\theta\\ y'=-\frac{\sqrt{-f}}{\sqrt{a}\cos t}\sin\theta+\frac{\sqrt{-f}\tan t}{\sqrt{-c}}\cos\theta+p\sin\theta-q\cos\theta \end{cases}$$

Then we have

$$\begin{cases} x_1=\frac{\sqrt{-f}}{\sqrt{a}}\frac{\cos\theta}{\sin t_1}+\frac{\sqrt{-f}}{\sqrt{-c}}\tan t_1\sin\theta-p\cos\theta-q\sin\theta\\ y_1=-\frac{\sqrt{-f}}{\sqrt{a}}\frac{\sin\theta}{\cos t_1}+\frac{\sqrt{-f}}{\sqrt{-c}}\tan t_1\cos\theta+p\sin\theta-q\cos\theta \end{cases}$$

$$\left\{\begin{array}{l} x_2 = \frac{\sqrt{-f}}{\sqrt{a}}\frac{\cos\theta}{\sin t_2} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_2 \sin\theta - p\cos\theta - q\sin\theta \\ y_2 = -\frac{\sqrt{-f}}{\sqrt{a}}\frac{\sin\theta}{\cos t_2} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_2 \cos\theta + p\sin\theta - q\cos\theta \end{array}\right.$$

$$\left\{\begin{array}{l} x_3 = \frac{\sqrt{-f}}{\sqrt{a}}\frac{\cos\theta}{\sin t_3} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_3 \sin\theta - p\cos\theta - q\sin\theta \\ y_3 = -\frac{\sqrt{-f}}{\sqrt{a}}\frac{\sin\theta}{\cos t_3} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_3 \cos\theta + p\sin\theta - q\cos\theta \end{array}\right.$$

$$\left\{\begin{array}{l} m = ((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\frac{1}{\sin t_3} - \frac{1}{\sin t_2}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_2))^2 + (-\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\frac{1}{\cos t_3} - \frac{1}{\cos t_2}) \\ +\frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_2))^2)^{\frac{1}{2}} \\ n = ((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\frac{1}{\sin t_3} - \frac{1}{\sin t_1}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_1))^2 + (-\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\frac{1}{\cos t_3} - \frac{1}{\cos t_1}) \\ +\frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_1))^2)^{\frac{1}{2}} \\ q = ((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\frac{1}{\sin t_2} - \frac{1}{\sin t_1}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_2 - \tan t_1))^2 + (-\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\frac{1}{\cos t_2} - \frac{1}{\cos t_1}) \\ +\frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_2 - \tan t_1))^2)^{\frac{1}{2}} \end{array}\right.$$

Then we have

$$\left\{\begin{array}{l} x_I = (((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\frac{1}{\sin t_3} - \frac{1}{\sin t_2}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_2))^2 + (-\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\frac{1}{\cos t_3} - \frac{1}{\cos t_2}) \\ +\frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_2))^2)^{\frac{1}{2}}(\frac{\sqrt{-f}}{\sqrt{a}}\frac{\cos\theta}{\sin t_1} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_1\sin\theta - p\cos\theta - q\sin\theta) + \\ ((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\frac{1}{\sin t_3} - \frac{1}{\sin t_1}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_1))^2 + (-\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\frac{1}{\cos t_3} - \frac{1}{\cos t_1}) + \\ \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_1))^2)^{\frac{1}{2}}(\frac{\sqrt{-f}}{\sqrt{a}}\frac{\cos\theta}{\sin t_2} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_2\sin\theta - p\cos\theta - q\sin\theta) \\ +((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\frac{1}{\sin t_2} - \frac{1}{\sin t_1}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_2 - \tan t_1))^2 + (-\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\frac{1}{\cos t_2} - \frac{1}{\cos t_1}) \\ +\frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_2 - \tan t_1))^2)^{\frac{1}{2}}(\frac{\sqrt{-f}}{\sqrt{a}}\frac{\cos\theta}{\sin t_3} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_3\sin\theta - p\cos\theta - q\sin\theta))/ \\ (((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\frac{1}{\sin t_3} - \frac{1}{\sin t_2}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_2))^2 + (-\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\frac{1}{\cos t_3} - \frac{1}{\cos t_2}) \\ +\frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_2))^2)^{\frac{1}{2}} + ((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\frac{1}{\sin t_3} - \frac{1}{\sin t_1}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_1))^2 \\ +(-\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\frac{1}{\cos t_3} - \frac{1}{\cos t_1}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_1))^2)^{\frac{1}{2}} + ((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\frac{1}{\sin t_2} - \frac{1}{\sin t_1}) + \\ \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_2 - \tan t_1))^2 + (-\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\frac{1}{\cos t_2} - \frac{1}{\cos t_1}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_2 - \tan t_1))^2)^{\frac{1}{2}}) \\ \\ y_I = (((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\frac{1}{\sin t_3} - \frac{1}{\sin t_2}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_2))^2 + (-\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\frac{1}{\cos t_3} - \frac{1}{\cos t_2}) \\ +\frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_2))^2)^{\frac{1}{2}}(-\frac{\sqrt{-f}}{\sqrt{a}}\frac{\sin\theta}{\cos t_1} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_1\cos\theta + p\sin\theta - q\cos\theta) + \\ ((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\frac{1}{\sin t_3} - \frac{1}{\sin t_1}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_1))^2 + (-\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\frac{1}{\cos t_3} - \frac{1}{\cos t_1}) \\ +\frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_1))^2)^{\frac{1}{2}}(-\frac{\sqrt{-f}}{\sqrt{a}}\frac{\sin\theta}{\cos t_2} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_2\cos\theta + p\sin\theta - q\cos\theta) + \\ ((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\frac{1}{\sin t_2} - \frac{1}{\sin t_1}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_2 - \tan t_1))^2 + (-\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\frac{1}{\cos t_2} - \frac{1}{\cos t_1}) \\ +\frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_2 - \tan t_1))^2)^{\frac{1}{2}}(-\frac{\sqrt{-f}}{\sqrt{a}}\frac{\sin\theta}{\cos t_3} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_3\cos\theta + p\sin\theta - q\cos\theta))/ \\ (((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\frac{1}{\sin t_3} - \frac{1}{\sin t_2}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_2))^2 + (-\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\frac{1}{\cos t_3} - \frac{1}{\cos t_2}) \\ +\frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_2))^2)^{\frac{1}{2}} + ((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\frac{1}{\sin t_3} - \frac{1}{\sin t_1}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_1))^2 \\ +(-\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\frac{1}{\cos t_3} - \frac{1}{\cos t_1}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_3 - \tan t_1))^2)^{\frac{1}{2}} + ((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\frac{1}{\sin t_2} - \frac{1}{\sin t_1}) + \\ \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_2 - \tan t_1))^2 + (-\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\frac{1}{\cos t_2} - \frac{1}{\cos t_1}) + \frac{\sqrt{-f}}{\sqrt{-c}}\cos\theta(\tan t_2 - \tan t_1))^2)^{\frac{1}{2}}) \end{array}\right.$$

Parabola We can now write "original" function for the parabola

$$\left\{\begin{array}{l} x = at^2 \\ y = at \end{array}\right.$$

Then we have

$$\left\{\begin{array}{l} x' = at^2\cos\theta + at\sin\theta - p\cos\theta - q\sin\theta \\ y' = -at^2\sin\theta + at\cos\theta + p\sin\theta - q\cos\theta \end{array}\right.$$

Then we have

$$\begin{cases} x_1 = at_1^2\cos\theta + at_1\sin\theta - p\cos\theta - q\sin\theta \\ y_1 = -at_1^2\sin\theta + at_1\cos\theta + p\sin\theta - q\cos\theta \end{cases}$$

$$\begin{cases} x_2 = at_2^2\cos\theta + at_2\sin\theta - p\cos\theta - q\sin\theta \\ y_2 = -at_2^2\sin\theta + at_2\cos\theta + p\sin\theta - q\cos\theta \end{cases}$$

$$\begin{cases} x_3 = at_3^2\cos\theta + at_3\sin\theta - p\cos\theta - q\sin\theta \\ y_3 = -at_3^2\sin\theta + at_3\cos\theta + p\sin\theta - q\cos\theta \end{cases}$$

$$\begin{cases} m = ((a\cos\theta(t_3-t_2)(t_3+t_2) + a\sin\theta(t_3-t_2))^2 + (-a\sin\theta(t_3-t_2)(t_3+t_2) + a\sin\theta(t_3-t_2)))^{\frac{1}{2}} \\ n = ((a\cos\theta(t_3-t_1)(t_3+t_1) + a\sin\theta(t_3-t_1))^2 + (-a\sin\theta(t_3-t_1)(t_3+t_1) + a\sin\theta(t_3-t_1)))^{\frac{1}{2}} \\ q = ((a\cos\theta(t_2-t_1)(t_2+t_1) + a\sin\theta(t_2-t_1))^2 + (-a\sin\theta(t_2-t_1)(t_2+t_1) + a\sin\theta(t_2-t_1)))^{\frac{1}{2}} \end{cases}$$

Then we have

$$\begin{cases} x_I = (((a\cos\theta(t_3-t_2)(t_3+t_2) + a\sin\theta(t_3-t_2))^2 + (-a\sin\theta(t_3-t_2)(t_3+t_2) + a\sin\theta(t_3-t_2)))^{\frac{1}{2}} \\ (at_1^2\cos\theta + at_1\sin\theta - p\cos\theta - q\sin\theta) + ((a\cos\theta(t_3-t_1)(t_3+t_1) + a\sin\theta(t_3-t_1))^2 \\ +(-a\sin\theta(t_3-t_1)(t_3+t_1) + a\sin\theta(t_3-t_1)))^{\frac{1}{2}}(at_2^2\cos\theta + at_2\sin\theta - p\cos\theta - q\sin\theta) + \\ ((a\cos\theta(t_2-t_1)(t_2+t_1) + a\sin\theta(t_2-t_1))^2 + (-a\sin\theta(t_2-t_1)(t_2+t_1) + a\sin\theta(t_2-t_1)))^{\frac{1}{2}} \\ at_3^2\cos\theta + at_3\sin\theta - p\cos\theta - q\sin\theta)/(((a\cos\theta(t_3-t_2)(t_3+t_2) + asin\theta(t_3-t_2))^2 \\ +(-a\sin\theta(t_3-t_2)(t_3+t_2) + a\sin\theta(t_3-t_2)))^{\frac{1}{2}} + ((a\cos\theta(t_3-t_1)(t_3+t_1) + a\sin\theta(t_3-t_1))^2 \\ +(-a\sin\theta(t_3-t_1)(t_3+t_1) + a\sin\theta(t_3-t_1)))^{\frac{1}{2}} + ((a\cos\theta(t_2-t_1)(t_2+t_1) + a\sin\theta(t_2-t_1))^2 + \\ (-a\sin\theta(t_2-t_1)(t_2+t_1) + a\sin\theta(t_2-t_1)))^{\frac{1}{2}}) \\ \\ y_I = (((a\cos\theta(t_3-t_2)(t_3+t_2) + a\sin\theta(t_3-t_2))^2 + (-a\sin\theta(t_3-t_2)(t_3+t_2) + a\sin\theta(t_3-t_2)))^{\frac{1}{2}} \\ (-at_1^2\sin\theta + at_1\cos\theta + p\sin\theta - q\cos\theta) + ((a\cos\theta(t_3-t_1)(t_3+t_1) + a\sin\theta(t_3-t_1))^2 \\ +(-a\sin\theta(t_3-t_1)(t_3+t_1) + a\sin\theta(t_3-t_1)))^{\frac{1}{2}}(-at_2^2\sin\theta + at_2\cos\theta + p\sin\theta - q\cos\theta) + \\ ((a\cos\theta(t_2-t_1)(t_2+t_1) + a\sin\theta(t_2-t_1))^2 + (-a\sin\theta(t_2-t_1)(t_2+t_1) + a\sin\theta(t_2-t_1)))^{\frac{1}{2}} \\ (-at_3^2\sin\theta + at_3\cos\theta + p\sin\theta - q\cos\theta))/(((a\cos\theta(t_3-t_2)(t_3+t_2) + a\sin\theta(t_3-t_2))^2 \\ +(-a\sin\theta(t_3-t_2)(t_3+t_2) + a\sin\theta(t_3-t_2)))^{\frac{1}{2}} + ((a\cos\theta(t_3-t_1)(t_3+t_1) + a\sin\theta(t_3-t_1))^2 \\ +(-a\sin\theta(t_3-t_1)(t_3+t_1) + a\sin\theta(t_3-t_1)))^{\frac{1}{2}} + ((a\cos\theta(t_2-t_1)(t_2+t_1) + a\sin\theta(t_2-t_1))^2 + \\ (-a\sin\theta(t_2-t_1)(t_2+t_1) + a\sin\theta(t_2-t_1)))^{\frac{1}{2}}) \end{cases}$$

From this we can know that in each kind of situation, we have different kinds of algebraic equations. The most important thing for us is that we get the algebraic and parametric expression with four different variables.

2.4.3 Othocentre's locus

Ellipse We have

$$\begin{cases} x = \frac{\sqrt{-f}}{\sqrt{a}}\sin t \\ y = \frac{\sqrt{-f}}{\sqrt{c}}\cos t \end{cases}$$

we now can get the prametric equation for the normal Ellipse $ax^2+bxy+cy^2+dx+ey+f=0$,that means

$$\begin{cases} x = \frac{\sqrt{-f}}{\sqrt{a}}\sin t\cos\theta + \frac{\sqrt{-f}}{\sqrt{a}}\sin\theta - v\cos\theta - w\sin\theta \\ y = -\frac{\sqrt{-f}}{\sqrt{c}}\sin t\sin\theta + \frac{\sqrt{-f}}{\sqrt{c}}\cos t\cos\theta + v\sin\theta - w\cos\theta \end{cases}$$

So that we have

$$\begin{cases} x_1 = \frac{\sqrt{-f}}{\sqrt{a}}\sin t_1\cos\theta + \frac{\sqrt{-f}}{\sqrt{a}}\cos t_1\sin\theta - v\cos\theta - w\sin\theta \\ y_1 = -\frac{\sqrt{-f}}{\sqrt{c}}\sin t_1\sin\theta + \frac{\sqrt{-f}}{\sqrt{c}}\cos t_1\cos\theta + v\sin\theta - w\cos\theta \end{cases}$$

$$\begin{cases} x_2 = \frac{\sqrt{-f}}{\sqrt{a}} \sin t_2 \cos\theta + \frac{\sqrt{-f}}{\sqrt{a}} \cos t_2 \sin\theta - v\cos\theta - w\sin\theta \\ y_2 = -\frac{\sqrt{-f}}{\sqrt{c}} \sin t_2 \sin\theta + \frac{\sqrt{-f}}{\sqrt{c}} \cos t_2 \cos\theta + v\sin\theta - w\cos\theta \end{cases}$$

$$\begin{cases} x_3 = \frac{\sqrt{-f}}{\sqrt{a}} \sin t_3 \cos\theta + \frac{\sqrt{-f}}{\sqrt{a}} \cos t_3 \sin\theta - v\cos\theta - w\sin\theta \\ y_3 = -\frac{\sqrt{-f}}{\sqrt{c}} \sin t_3 \sin\theta + \frac{\sqrt{-f}}{\sqrt{c}} \cos t_3 \cos\theta + v\sin\theta - w\cos\theta \end{cases}$$

$$\begin{cases}
\tan A = ((-\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta(\sin t_3 - \sin t_1) + \frac{\sqrt{-f}}{\sqrt{c}} \cos\theta(\cos t_3 - \cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}} \cos\theta(\sin t_2 - \sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{a}} \sin\theta(\cos t_2 - \cos t_1)) - (-\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta(\sin t_2 - \sin t_1) + \frac{\sqrt{-f}}{\sqrt{c}} \cos\theta(\cos t_2 - \cos t_1)) \\
(\frac{\sqrt{-f}}{\sqrt{a}} \cos\theta(\sin t_3 - \sin t_1) + \frac{\sqrt{-f}}{\sqrt{a}} \sin\theta(\cos t_3 - \cos t_1)))/((\frac{\sqrt{-f}}{\sqrt{a}} \cos\theta(\sin t_3 - \sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{a}} \sin\theta(\cos t_3 - \cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}} \cos\theta(\sin t_2 - \sin t_1) + \frac{\sqrt{-f}}{\sqrt{a}} \sin\theta(\cos t_2 - \cos t_1))+ \\
(-\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta(\sin t_3 - \sin t_1) + \frac{\sqrt{-f}}{\sqrt{c}} \cos\theta(\cos t_3 - \cos t_1))(-\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta(\sin t_2 - \sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{c}} \cos\theta(\cos t_2 - \cos t_1))) \\
\tan B = ((-\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta(\sin t_3 - \sin t_2) + \frac{\sqrt{-f}}{\sqrt{c}} \cos\theta(\cos t_3 - \cos t_2))(\frac{\sqrt{-f}}{\sqrt{a}} \cos\theta(\sin t_2 - \sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{a}} \sin\theta(\cos t_2 - \cos t_1)) - (-\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta(\sin t_2 - \sin t_1) + \frac{\sqrt{-f}}{\sqrt{c}} \cos\theta(\cos t_2 - \cos t_1)) \\
(\frac{\sqrt{-f}}{\sqrt{a}} \cos\theta(\sin t_3 - \sin t_2) + \frac{\sqrt{-f}}{\sqrt{a}} \sin\theta(\cos t_3 - \cos t_2)))/((\frac{\sqrt{-f}}{\sqrt{a}} \cos\theta(\sin t_3 - \sin t_2) \\
+\frac{\sqrt{-f}}{\sqrt{a}} \sin\theta(\cos t_3 - \cos t_2))(\frac{\sqrt{-f}}{\sqrt{a}} \cos\theta(\sin t_2 - \sin t_1) + \frac{\sqrt{-f}}{\sqrt{a}} \sin\theta(\cos t_2 - \cos t_1))+ \\
(-\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta(\sin t_3 - \sin t_2) + \frac{\sqrt{-f}}{\sqrt{c}} \cos\theta(\cos t_3 - \cos t_2))(-\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta(\sin t_2 - \sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{c}} \cos\theta(\cos t_2 - \cos t_1))) \\
\tan C = ((-\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta(\sin t_3 - \sin t_1) + \frac{\sqrt{-f}}{\sqrt{c}} \cos\theta(\cos t_3 - \cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}} \cos\theta(\sin t_3 - \sin t_2) \\
+\frac{\sqrt{-f}}{\sqrt{a}} \sin\theta(\cos t_3 - \cos t_2)) - (-\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta(\sin t_3 - \sin t_2) + \frac{\sqrt{-f}}{\sqrt{c}} \cos\theta(\cos t_3 - \cos t_2)) \\
(\frac{\sqrt{-f}}{\sqrt{a}} \cos\theta(\sin t_3 - \sin t_1) + \frac{\sqrt{-f}}{\sqrt{a}} \sin\theta(\cos t_3 - \cos t_1)))/((\frac{\sqrt{-f}}{\sqrt{a}} \cos\theta(\sin t_3 - \sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{a}} \sin\theta(\cos t_3 - \cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}} \cos\theta(\sin t_3 - \sin t_2) + \frac{\sqrt{-f}}{\sqrt{a}} \sin\theta(\cos t_3 - \cos t_2))+ \\
(-\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta(\sin t_3 - \sin t_1) + \frac{\sqrt{-f}}{\sqrt{c}} \cos\theta(\cos t_3 - \cos t_1))(-\frac{\sqrt{-f}}{\sqrt{c}} \sin\theta(\sin t_3 - \sin t_2) \\
+\frac{\sqrt{-f}}{\sqrt{c}} \cos\theta(\cos t_3 - \cos t_2)))
\end{cases}$$

Then we have

$$
\left\{
\begin{array}{l}
x_H = ((((-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))-(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1)) \\
(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1)))/((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))+ \\
(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1))))(\frac{\sqrt{-f}}{\sqrt{a}}\sin t_1\cos\theta+\frac{\sqrt{-f}}{\sqrt{a}}\cos t_1\sin\theta-v\cos\theta-w\sin\theta)+ \\
(((-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))-(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1)) \\
(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2)))/((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2) \\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))+ \\
(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1))))(\frac{\sqrt{-f}}{\sqrt{a}}\sin t_2\cos\theta+\frac{\sqrt{-f}}{\sqrt{a}}\cos t_2\sin\theta-v\cos\theta-w\sin\theta)+ \\
(((-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2) \\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))-(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2)) \\
(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1)))/((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))+ \\
(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2) \\
+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))))(\frac{\sqrt{-f}}{\sqrt{a}}\sin t_3\cos\theta+\frac{\sqrt{-f}}{\sqrt{a}}\cos t_3\sin\theta-v\cos\theta-w\sin\theta)/ \\
(((-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))-(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1)) \\
(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1)))/((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))+ \\
(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1)))+ \\
((-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))-(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1)) \\
(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2)))/((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2) \\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))+ \\
(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1)))+ \\
((-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2) \\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))-(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2)) \\
(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1)))/((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))+ \\
(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2) \\
+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2)))) \\
\\
y_H = ((((-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1) \\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))-(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1)) \\
(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1)))/((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)
\end{array}
\right.
$$

$$
\left\{
\begin{array}{l}
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))+\\
(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)\\
+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1))))(-\frac{\sqrt{-f}}{\sqrt{c}}\sin t_1\sin\theta+\frac{\sqrt{-f}}{\sqrt{c}}\cos t_1\cos\theta+v\sin\theta-w\cos\theta\\
)+(((-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)\\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))-(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1))\\
(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2)))/((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)\\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))+\\
(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)\\
+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1))))(-\frac{\sqrt{-f}}{\sqrt{c}}\sin t_2\sin\theta+\frac{\sqrt{-f}}{\sqrt{c}}\cos t_2\cos\theta+v\sin\theta-w\cos\theta)+\\
(((-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)\\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))-(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))\\
(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1)))/((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)\\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))+\\
(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)\\
+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))))(-\frac{\sqrt{-f}}{\sqrt{c}}\sin t_3\sin\theta+\frac{\sqrt{-f}}{\sqrt{c}}\cos t_3\cos\theta+v\sin\theta-w\cos\theta))/\\
(((-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)\\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))-(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1))\\
(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1)))/((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)\\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))+\\
(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)\\
+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1)))+\\
((-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)\\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))-(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1))\\
(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2)))/((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)\\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_2-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_2-\cos t_1))+\\
(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_2-\sin t_1)\\
+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_2-\cos t_1)))+\\
((-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)\\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))-(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))\\
(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1)))/((\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_1)\\
+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_1))(\frac{\sqrt{-f}}{\sqrt{a}}\cos\theta(\sin t_3-\sin t_2)+\frac{\sqrt{-f}}{\sqrt{a}}\sin\theta(\cos t_3-\cos t_2))+\\
(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_1)+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_1))(-\frac{\sqrt{-f}}{\sqrt{c}}\sin\theta(\sin t_3-\sin t_2)\\
+\frac{\sqrt{-f}}{\sqrt{c}}\cos\theta(\cos t_3-\cos t_2))))
\end{array}
\right.
$$

Hyperbola We have

$$
\left\{
\begin{array}{l}
x=\frac{\sqrt{-f}}{\sqrt{a}\cos t}\\
y=\frac{\sqrt{-f}\tan t}{\sqrt{-c}}
\end{array}
\right.
$$

Then we have

$$
\left\{
\begin{array}{l}
x_1=\frac{\sqrt{-f}}{\sqrt{a}}\frac{\cos\theta}{\sin t_1}+\frac{\sqrt{-f}}{\sqrt{-c}}\tan t_1\sin\theta-p\cos\theta-q\sin\theta\\
y_1=-\frac{\sqrt{-f}}{\sqrt{a}}\frac{\sin\theta}{\cos t_1}+\frac{\sqrt{-f}}{\sqrt{-c}}\tan t_1\cos\theta+p\sin\theta-q\cos\theta
\end{array}
\right.
$$

$$\begin{cases} x_2 = \frac{\sqrt{-f}}{\sqrt{a}}\frac{\cos\theta}{\sin t_2} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_2 \sin\theta - p\cos\theta - q\sin\theta \\ y_2 = -\frac{\sqrt{-f}}{\sqrt{a}}\frac{\sin\theta}{\cos t_2} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_2 \cos\theta + p\sin\theta - q\cos\theta \end{cases}$$

$$\begin{cases} x_3 = \frac{\sqrt{-f}}{\sqrt{a}}\frac{\cos\theta}{\sin t_3} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_3 \sin\theta - p\cos\theta - q\sin\theta \\ y_3 = -\frac{\sqrt{-f}}{\sqrt{a}}\frac{\sin\theta}{\cos t_3} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_3 \cos\theta + p\sin\theta - q\cos\theta \end{cases}$$

We now just need to replace the $x_1, x_2, x_3, y_1, y_2, y_3$ (the Ellipse situation for Othocentre's locus)with the $x_1, x_2, x_3, y_1, y_2, y_3$ above.

Parabola We have

$$\begin{cases} x = at^2 \\ y = at \end{cases}$$

Then we have

$$\begin{cases} x_1 = at_1^2\cos\theta + at_1\sin\theta - p\cos\theta - q\sin\theta \\ y_1 = -at_1^2\sin\theta + at_1\cos\theta + p\sin\theta - q\cos\theta \end{cases}$$

$$\begin{cases} x_2 = at_2^2\cos\theta + at_2\sin\theta - p\cos\theta - q\sin\theta \\ y_2 = -at_2^2\sin\theta + at_2\cos\theta + p\sin\theta - q\cos\theta \end{cases}$$

$$\begin{cases} x_3 = at_3^2\cos\theta + at_3\sin\theta - p\cos\theta - q\sin\theta \\ y_3 = -at_3^2\sin\theta + at_3\cos\theta + p\sin\theta - q\cos\theta \end{cases}$$

We now just need to replace the $x_1, x_2, x_3, y_1, y_2, y_3$ (the Ellipse situation for Othocentre's locus) with the $x_1, x_2, x_3, y_1, y_2, y_3$ above.

2.4.4 Intersection of symmedian's locus

$$\begin{cases} x = \frac{x_1+x_2+x_3+(\frac{a^2}{c^2}+\frac{a^2}{b^2})x_1+(\frac{b^2}{a^2}+\frac{b^2}{c^2})x_2+(\frac{c^2}{a^2}+\frac{c^2}{b^2})x_3}{a^2+b^2+c^2} \\ y = \frac{y_1+y_2+y_3+(\frac{a^2}{c^2}+\frac{a^2}{b^2})y_1+(\frac{b^2}{a^2}+\frac{b^2}{c^2})y_2+(\frac{c^2}{a^2}+\frac{c^2}{b^2})y_3}{a^2+b^2+c^2} \end{cases} \quad (here\quad a,b\quad c\quad are\quad the\quad length\quad of\quad the\quad triangle)$$

Ellipse

$$\begin{cases} x_1 = \frac{\sqrt{-f}}{\sqrt{a}}\sin t_1\cos\theta + \frac{\sqrt{-f}}{\sqrt{a}}\cos t_1\sin\theta - v\cos\theta - w\sin\theta \\ y_1 = -\frac{\sqrt{-f}}{\sqrt{c}}\sin t_1\sin\theta + \frac{\sqrt{-f}}{\sqrt{c}}\cos t_1\cos\theta + v\sin\theta - w\cos\theta \end{cases}$$

$$\begin{cases} x_2 = \frac{\sqrt{-f}}{\sqrt{a}}\sin t_2\cos\theta + \frac{\sqrt{-f}}{\sqrt{a}}\cos t_2\sin\theta - v\cos\theta - w\sin\theta \\ y_2 = -\frac{\sqrt{-f}}{\sqrt{c}}\sin t_2\sin\theta + \frac{\sqrt{-f}}{\sqrt{c}}\cos t_2\cos\theta + v\sin\theta - w\cos\theta \end{cases}$$

$$\begin{cases} x_3 = \frac{\sqrt{-f}}{\sqrt{a}}\sin t_3\cos\theta + \frac{\sqrt{-f}}{\sqrt{a}}\cos t_3\sin\theta - v\cos\theta - w\sin\theta \\ y_3 = -\frac{\sqrt{-f}}{\sqrt{c}}\sin t_3\sin\theta + \frac{\sqrt{-f}}{\sqrt{c}}\cos t_3\cos\theta + v\sin\theta - w\cos\theta \end{cases}$$

$$\begin{cases} a = ((a\cos\theta(t_3-t_2)(t_3+t_2) + asin\theta(t_3-t_2))^2 + (-a\sin\theta(t_3-t_2)(t_3+t_2) + a\sin\theta(t_3-t_2)))^{\frac{1}{2}} \\ b = ((a\cos\theta(t_3-t_1)(t_3+t_1) + asin\theta(t_3-t_1))^2 + (-a\sin\theta(t_3-t_1)(t_3+t_1) + a\sin\theta(t_3-t_1)))^{\frac{1}{2}} \\ c = ((a\cos\theta(t_2-t_1)(t_2+t_1) + asin\theta(t_2-t_1))^2 + (-a\sin\theta(t_2-t_1)(t_2+t_1) + a\sin\theta(t_2-t_1)))^{\frac{1}{2}} \end{cases}$$

We just need to replace the things above, we can get the function.

Hyperbola

$$\begin{cases} x_1 = \frac{\sqrt{-f}}{\sqrt{a}}\frac{\cos\theta}{\sin t_1} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_1 \sin\theta - p\cos\theta - q\sin\theta \\ y_1 = -\frac{\sqrt{-f}}{\sqrt{a}}\frac{\sin\theta}{\cos t_1} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_1 \cos\theta + p\sin\theta - q\cos\theta \end{cases}$$

$$\begin{cases} x_2 = \frac{\sqrt{-f}}{\sqrt{a}}\frac{\cos\theta}{\sin t_2} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_2 \sin\theta - p\cos\theta - q\sin\theta \\ y_2 = -\frac{\sqrt{-f}}{\sqrt{a}}\frac{\sin\theta}{\cos t_2} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_2 \cos\theta + p\sin\theta - q\cos\theta \end{cases}$$

$$\begin{cases} x_3 = \frac{\sqrt{-f}}{\sqrt{a}}\frac{\cos\theta}{\sin t_3} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_3 \sin\theta - p\cos\theta - q\sin\theta \\ y_3 = -\frac{\sqrt{-f}}{\sqrt{a}}\frac{\sin\theta}{\cos t_3} + \frac{\sqrt{-f}}{\sqrt{-c}}\tan t_3 \cos\theta + p\sin\theta - q\cos\theta \end{cases}$$

$$\begin{cases} a = ((a\cos\theta(t_3-t_2)(t_3+t_2) + a\sin\theta(t_3-t_2))^2 + (-a\sin\theta(t_3-t_2)(t_3+t_2) + a\sin\theta(t_3-t_2)))^{\frac{1}{2}} \\ b = ((a\cos\theta(t_3-t_1)(t_3+t_1) + a\sin\theta(t_3-t_1))^2 + (-a\sin\theta(t_3-t_1)(t_3+t_1) + a\sin\theta(t_3-t_1)))^{\frac{1}{2}} \\ c = ((a\cos\theta(t_2-t_1)(t_2+t_1) + a\sin\theta(t_2-t_1))^2 + (-a\sin\theta(t_2-t_1)(t_2+t_1) + a\sin\theta(t_2-t_1)))^{\frac{1}{2}} \end{cases}$$

We just need to replace the function above, we can get the function.

Parabola

$$\begin{cases} x_1 = \frac{\sqrt{-f}}{\sqrt{a}}\sin t_1\cos\theta + \frac{\sqrt{-f}}{\sqrt{a}}\cos t_1 \sin\theta - v\cos\theta - w\sin\theta \\ y_1 = -\frac{\sqrt{-f}}{\sqrt{c}}\sin t_1\sin\theta + \frac{\sqrt{-f}}{\sqrt{c}}\cos t_1 \cos\theta + v\sin\theta - w\cos\theta \end{cases}$$

$$\begin{cases} x_2 = \frac{\sqrt{-f}}{\sqrt{a}}\sin t_2\cos\theta + \frac{\sqrt{-f}}{\sqrt{a}}\cos t_2 \sin\theta - v\cos\theta - w\sin\theta \\ y_2 = -\frac{\sqrt{-f}}{\sqrt{c}}\sin t_2\sin\theta + \frac{\sqrt{-f}}{\sqrt{c}}\cos t_2 \cos\theta + v\sin\theta - w\cos\theta \end{cases}$$

$$\begin{cases} x_3 = \frac{\sqrt{-f}}{\sqrt{a}}\sin t_3\cos\theta + \frac{\sqrt{-f}}{\sqrt{a}}\cos t_3 \sin\theta - v\cos\theta - w\sin\theta \\ y_3 = -\frac{\sqrt{-f}}{\sqrt{c}}\sin t_3\sin\theta + \frac{\sqrt{-f}}{\sqrt{c}}\cos t_3 \cos\theta + v\sin\theta - w\cos\theta \end{cases}$$

$$\begin{cases} a = ((a\cos\theta(t_3-t_2)(t_3+t_2) + a\sin\theta(t_3-t_2))^2 + (-a\sin\theta(t_3-t_2)(t_3+t_2) + a\sin\theta(t_3-t_2)))^{\frac{1}{2}} \\ b = ((a\cos\theta(t_3-t_1)(t_3+t_1) + a\sin\theta(t_3-t_1))^2 + (-a\sin\theta(t_3-t_1)(t_3+t_1) + a\sin\theta(t_3-t_1)))^{\frac{1}{2}} \\ c = ((a\cos\theta(t_2-t_1)(t_2+t_1) + a\sin\theta(t_2-t_1))^2 + (-a\sin\theta(t_2-t_1)(t_2+t_1) + a\sin\theta(t_2-t_1)))^{\frac{1}{2}} \end{cases}$$

We just need to replace the function above, we can get the function

2.5 Summarize

2.5.1 Incentre's locus

The locus actually is made up by two parts and when $ax^2 + bxy + cy^2 + dx + ey + f = 0$ are different kinds of second-order curve, we have different situations

Ellipse when three points are in a Ellipse,one is under the line AB, the other is over the line AB, and they both a part of a Ellipse,A and B are break points

Hyperbola when it is a hyperbola, its two parts are both part of hyperbola. When A,B's y-arex value are both positiv or negativ and are not on the same branch of the hyperbola, the two hyperbola are "parallel symmetry"(the symmetry axes of two hyperbolas are parallel) if they are not in the same positivity or negativity, the two hyperbola are "cross symmetry"(the symmetry axes of two hyperbolas are not parallel), and B are break points

Parabola When it is a parabola, no matter A,B are on the both sides or only one side of the symmetry axe,there is one hyperbola over the line AB und one below the line AB.A and B are break points

2.5.2 Orthocentre's locus

there are two situations, ein is non-right triangle and the another is right triangle

Non-right triangle

Ellipse When it is a Ellipse, the graph is the same as $a(\frac{x}{3})^2 + b(\frac{x}{3})(\frac{y}{3}) + c(\frac{y}{3})^2 + d(\frac{x}{3}) + e(\frac{y}{3}) + f = 0$, it is just a graph of $a(\frac{x}{3})^2 + b(\frac{x}{3})(\frac{y}{3}) + c(\frac{y}{3})^2 + d(\frac{x}{3}) + e(\frac{y}{3}) + f = 0$ and move it randomly.

Hyperbola When it is a hyperbola, the graph is the same as $a(\frac{x}{3})^2 + b(\frac{x}{3})(\frac{y}{3}) + c(\frac{y}{3})^2 + d(\frac{x}{3}) + e(\frac{y}{3}) + f = 0$, it is just a graph of $a(\frac{x}{3})^2 + b(\frac{x}{3})(\frac{y}{3}) + c(\frac{y}{3})^2 + d(\frac{x}{3}) + e(\frac{y}{3}) + f = 0$ and move it randomly.

Parabola When it is a Parabola, when we use $x = ay^2$, this type to deschreibe the algebraic expression of the locus, it has a same graph with $x = 3ay^2$

Right triangle

Ellipse When it is a Ellipse, the graph is the same as $a(\frac{x}{6})^2 + b(\frac{x}{6})(\frac{y}{6}) + c(\frac{y}{6})^2 + d(\frac{x}{6}) + e(\frac{y}{6}) + f = 0$, it is just a graph of $a(\frac{x}{6})^2 + b(\frac{x}{6})(\frac{y}{6}) + c(\frac{y}{6})^2 + d(\frac{x}{6}) + e(\frac{y}{6}) + f = 0$m , and move it randomly.

Hyperbola When it is a hyperbola, the graph is the same as $ax^2+bxy+cy^2+dx+ey+f = 0$, it is just a graph of $ax^2 + bxy + cy^2 + dx + ey + f = 0$ and move it randomly.

Parabola When it is a Parabola, when we use $x = ay^2$, this type to deschreibe the algebraic expression of the locus, it has a same graph with $x = ay^2$

2.5.3 Circumcentre's locus

The locus of circumcentre's locus is a second-order curve

$$y = -\frac{x_1 - x_2}{y_1 - y_2}x + \frac{x_1^2 + y_1^2 - x_2^2 - y_2^2}{2y_1 - 2y_2}$$

2.5.4 intersection of medians' locus

The locus of intersection of median's locus is a second-order curve

$$a(3x - x_1 - x_2)^2 + b(3x - x_1 - x_2)(3y - y_1 - y_2)+$$

$$c(3y - y_1 - y_2)^2 + d(3x - x_1 - x_2) + e(3y - y_1 - y_2) + f = 0$$

2.5.5 intersection of symmedians' locus

Ellipse When it is a Ellipse,three is three different situations, three are two different kind of "inward depression", one is the "inward depression" on the sides and the another is the "inward depression" in the middle

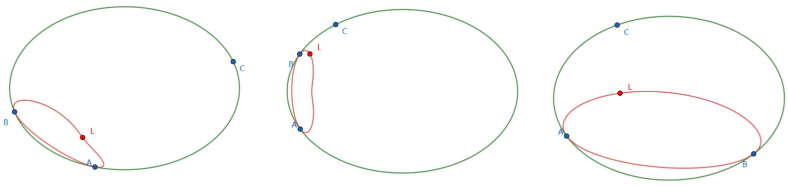

Figure 4: Three different situations

Hyperbola When it is a hyperbola, it is a very complex graph, the graph is made up with two parts,it is to difficult to deschreibe with language.If the two points are on different branchs,we have

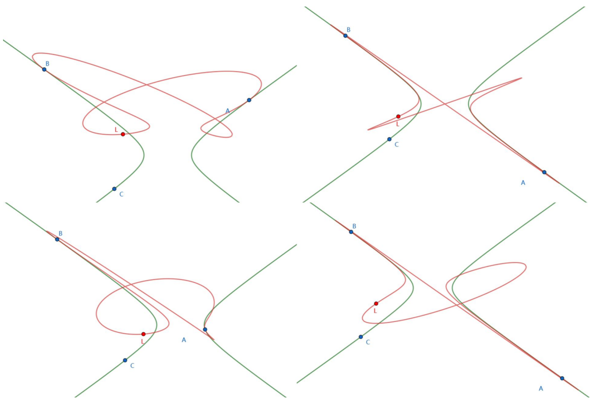

Figure 5: Four situations on two branches of he function

If the two points are on the same sides, we have

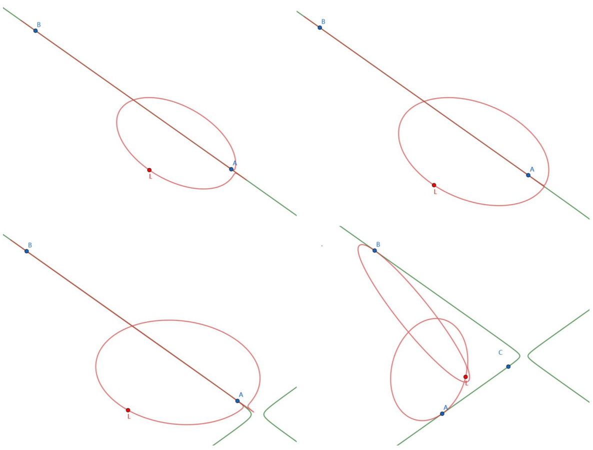

Figure 6: Four situations on one sanme branches of the function

There are some differences in the shape of the, in the first picture, the tip is under the line, and in the second picture, there are no tips, and in the third picture, the tip is above the line

Parabola There are three different graphs in this kind of situation, two of them are same type, their two points A,B are on the different sides of the axis of symmetry, and another one is one side of the axis of symmetry.

If the two points A,B are on the different sides of the axis of symmetry, then we have

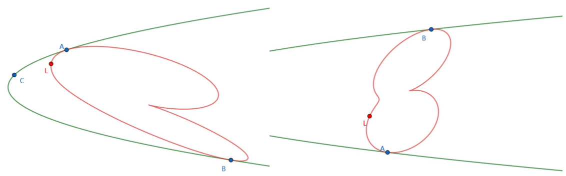

Figure 7: A,B are on the different sides of the axis of symmetry

If the two points A,B are on the same side of the axis of symmetry, then we have

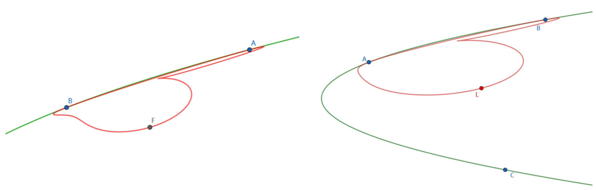

Figure 8: A,B are on the same sides of the axis of symmetry

2.5.6 Suggest and study additional directions of research

In future research, we can use graph theory to illustrate that trajectories do not actually have intersections, which means that some areas that appear to have a intersection actually do not have any intersections. At the same time, we can also use Fourier transform to fit overly complex equations. What we need to examine clearly is the relationship between various coefficients and the coordinates of triangle vertices.

3 Literature

1. Shen Wenxuan;Yang Qingtao,New Analysis of Graphic Characteristics [M] Harbin, Harbin Institute of Technology Press,2019,202-204