The Picard Homotopy Type of a Derived Smooth Manifold

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The question

Can one construct an intrinsic Brauer-type object for a derived smooth manifold which retains the higher homotopy of its structure sheaf?

From smooth rings to spectra

Let $X = (|X|, \mathcal{O}_X)$ be a derived smooth manifold in the sense of Spivak.

For a simplicial C^∞-ring A, forget the non-polynomial smooth operations:
$$A \longmapsto A^{\mathrm{alg}}.$$
Over $\mathbb{R}$ there are equivalences
$$s\mathrm{CAlg}_{\mathbb{R}}[W^{-1}] \simeq \mathrm{cdga}_{\mathbb{R}}^{\geq 0}[W^{-1}] \simeq \mathrm{CAlg}_{H\mathbb{R}}^{\mathrm{cn}}.$$
Set
$$A^{\mathrm{sp}} := \Phi\Psi(A^{\mathrm{alg}}).$$
After hypersheafification this produces a connective commutative ring-spectrum sheaf $\mathbb{O}_X$.

For every $x \in |X|$,
$$\mathbb{O}_{X,x} \simeq \Phi\Psi(\mathcal{O}_{X,x}^{\mathrm{alg}}).$$
Since $\pi_0(\mathcal{O}_{X,x}^{\mathrm{alg}})$ is local,
$$\boxed{\mathbb{O}_{X,x} \text{ is a connective local } E_\infty\text{-ring.}}$$

Intrinsic module categories

Let
$$\mathcal{X} := \mathrm{Shv}(|X|, \mathcal{S})^{\wedge}.$$
For $U \subseteq |X|$, define the intrinsic module category
$$\boxed{M(U) := \mathrm{Mod}_{\mathbb{O}_U}(\mathrm{Sp}(\mathcal{X}_{/U})).}$$
It depends on the full ring-spectrum sheaf $\mathbb{O}_U$, not only on $\Gamma(U, \mathbb{O}_U)$.

For every open hypercover $U_\bullet \to U$,
$$\boxed{M(U) \simeq \lim_{[n]\in\Delta} M(U_n).}$$
Hence
$$\mathrm{Pic}_{\mathbb{O}_X}(U) := \mathrm{Pic}(M(U))$$
is a hypersheaf of grouplike E_∞-spaces, with unit hypersheaf
$$\mathrm{GL}_1(\mathbb{O}_X)(U) = \mathrm{Aut}_{M(U)}(\mathbb{O}_U).$$
For connective R,
$$\pi_0 \mathrm{GL}_1(R) = (\pi_0 R)^\times, \qquad \pi_i \mathrm{GL}_1(R) = \pi_i R \quad (i \geq 1),$$
so the derived unit space retains the positive homotopy of the structure sheaf.

Construction and homotopy type

$$\begin{tikzcd}
\begin{array}{c}\text{derived smooth}\\ \text{manifold}\\ (|X|, \mathcal{O}_X)\end{array} \arrow[r] & \begin{array}{c}\text{spectralization}\\ \mathcal{O}_X \rightsquigarrow \mathbb{O}_X\end{array} \arrow[r] & \begin{array}{c}\text{intrinsic modules}\\ M(U)\end{array} \arrow[d] \\
& \begin{array}{c}\text{Picard hyper-}\\ \text{sheaf } \mathrm{Pic}_{\mathbb{O}_X}\end{array} \arrow[r] & K(\mathbb{Z},1) \times B^2\mathrm{GL}_1(\mathbb{O}_X)
\end{tikzcd}$$

The key local computation

Taking stalks gives
$$\theta_x : (\mathrm{Pic}_{\mathbb{O}_X})_x \longrightarrow \mathrm{Pic}(\mathbb{O}_{X,x}),$$
and θ_x is an equivalence.

For a connective local E_∞-ring R, Antieau–Gepner, Theorem 7.9, gives
$$\pi_0 \mathrm{Pic}(R) \cong \mathbb{Z}, \qquad n \longleftrightarrow \Sigma^n R.$$
Moreover,
$$\mathrm{Aut}_{\mathrm{Mod}_R}(\Sigma^n R) \simeq \mathrm{GL}_1(R).$$
Thus every component of $\mathrm{Pic}(R)$ is a copy of $B\,\mathrm{GL}_1(R)$:
$$\boxed{\mathrm{Pic}(R) \simeq \mathbb{Z} \times B\,\mathrm{GL}_1(R).}$$
Consequently,
$$(\mathrm{Pic}_{\mathbb{O}_X})_x \simeq \mathbb{Z} \times B\,\mathrm{GL}_1(\mathbb{O}_{X,x}).$$

The Picard–Brauer theorem

Suspension gives
$$\deg : \mathrm{Pic}_{\mathbb{O}_X} \longrightarrow \mathbb{Z}.$$
Its fibre is $B\,\mathrm{GL}_1(\mathbb{O}_X)$, and the suspension section splits the extension. Hence
$$\mathrm{Pic}_{\mathbb{O}_X} \simeq \mathbb{Z} \times B\,\mathrm{GL}_1(\mathbb{O}_X)$$
as hypersheaves of grouplike E_1-spaces.

Define
$$\mathrm{Br}_X^{\mathrm{Pic}} := B\,\mathrm{Pic}_{\mathbb{O}_X}.$$
Then there is a natural equivalence
$$\boxed{\mathrm{Br}_X^{\mathrm{Pic}} \simeq K(\mathbb{Z},1) \times B^2\mathrm{GL}_1(\mathbb{O}_X).}$$

What does it remember?

The homotopy sheaves begin with
$$\pi_1 \mathrm{Br}_X^{\mathrm{Pic}} = \mathbb{Z}, \qquad \pi_2 \mathrm{Br}_X^{\mathrm{Pic}} = (\pi_0\mathbb{O}_X)^\times,$$
while for $i \geq 3$,
$$\boxed{\pi_i \mathrm{Br}_X^{\mathrm{Pic}} \cong \pi_{i-2}\mathbb{O}_X.}$$
Thus the higher Postnikov layers record precisely the positive homotopy of the derived structure sheaf.

Global classes

Taking global sections gives a natural bijection of pointed sets
$$\pi_0\Gamma(X, \mathrm{Br}_X^{\mathrm{Pic}}) \cong H^1(X;\mathbb{Z}) \times \pi_0\Gamma(X, B^2\mathrm{GL}_1(\mathbb{O}_X))$$

Classical recovery

For a discrete commutative ring sheaf A with local stalks,
$$\mathrm{GL}_1(HA) = A^\times, \qquad \mathrm{Br}_{HA}^{\mathrm{Pic}} \simeq K(\mathbb{Z},1) \times K(A^\times, 2).$$
For an ordinary paracompact smooth manifold M:

Real coefficients. Since
$$C_M^\infty(\mathbb{R})^\times \simeq \underline{\mathbb{Z}/2} \times C_M^\infty(\mathbb{R})$$
and $C_M^\infty(\mathbb{R})$ is acyclic in positive degree,
$$\boxed{\pi_0\Gamma(M, \mathrm{Br}_{M,\mathbb{R}}^{\mathrm{Pic}}) \cong H^1_{\mathrm{sing}}(M;\mathbb{Z}) \times H^2_{\mathrm{sing}}(M;\mathbb{Z}/2).}$$
Complex coefficients. The smooth exponential sequence gives
$$H^2(M; C_M^\infty(\mathbb{C})^\times) \cong H^3_{\mathrm{sing}}(M;\mathbb{Z}),$$
and therefore
$$\boxed{\pi_0\Gamma(M, \mathrm{Br}_{M,\mathbb{C}}^{\mathrm{Pic}}) \cong H^1_{\mathrm{sing}}(M;\mathbb{Z}) \times H^3_{\mathrm{sing}}(M;\mathbb{Z}).}$$
The H^3 factor is the Dixmier–Douady contribution.

Warning. The real formula describes the ordinary open-site Picard-torsorial theory, not the full real Azumaya Brauer group.

Categorical interpretation

The Picard-torsor theorem is **unconditional**.

Assuming category-valued open hyperdescent,
$$\mathrm{Forms}_{\mathbb{O}_X} \simeq B\,\mathrm{Pic}_{\mathbb{O}_X} = \mathrm{Br}_X^{\mathrm{Pic}}.$$
Realization by an internal E_1-algebra requires additional compact-local-generator, base-change, and Morita hypotheses.

Thus no unconditional derived-Azumaya representability claim is made.

References

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